BITABLEAUX BASES FOR GARSIA-HAIMAN MODULES OF HOLLOW TYPE

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ABSTRACT. Garsia-Haiman modules $\mathbb{C}[X_n, Y_n]_{\mathcal{I}_\gamma}$ are quotient rings in variables $X_n = \{x_1, x_2, \ldots, x_n\}$ and $Y_n = \{y_1, y_2, \ldots, y_n\}$ that generalize the quotient ring $\mathbb{C}[X_n]/\mathcal{I}$, where $\mathcal{I}$ is the ideal generated by the elementary symmetric polynomials $e_j(X_n)$ for $1 \le j \le n$. A bitableau basis for the Garsia-Haiman modules of hollow type is constructed. Applications of this basis to representation theory and other related polynomial spaces are considered.

1. INTRODUCTION

Let $X_n = \{x_1, \ldots, x_n\}$ and $Y_n = \{y_1, \ldots, y_n\}$ be sets of indeterminates. The main purpose of this paper is to give explicit combinatorial bases for certain quotients of the ring

$$\mathbb{C}[X_n, Y_n] = \mathbb{C}[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n] \tag{1}$$

of polynomials in the variables X_n and Y_n with complex coefficients. In doing so, we give combinatorial interpretations for the corresponding Hilbert and Frobenius series. The ideals in the aforementioned quotients are defined via determinants as described below.

Throughout this paper, we will identify any element $\alpha_i = (\alpha_{i,1}, \alpha_{i,2}) \in \mathbb{N}^2$ with the unit square in the first quadrant of the plane having α_i as its corner closest to the origin. A *lattice diagram*, $L[\boldsymbol{\alpha}] = (\alpha_1, \ldots, \alpha_n)$, is a sequence of such unit squares. Writing $z_j^{\alpha_i}$ for the product $x_j^{\alpha_{i,1}} y_j^{\alpha_{i,2}}$, to any lattice diagram $L[\boldsymbol{\alpha}]$ we associate a determinant

$$\Delta_{L[\boldsymbol{\alpha}]} = \Delta_{L[\boldsymbol{\alpha}]}(X_n, Y_n) = \det \begin{pmatrix} z_1^{\alpha_1} & z_2^{\alpha_1} & \cdots & z_n^{\alpha_1} \\ z_1^{\alpha_2} & z_2^{\alpha_2} & \cdots & z_n^{\alpha_2} \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{\alpha_n} & z_2^{\alpha_n} & \cdots & z_n^{\alpha_n} \end{pmatrix}. \tag{2}$$

Given any polynomial $P(X_n, Y_n) \in \mathbb{C}[X_n, Y_n]$, there is a corresponding polynomial of differential operators

$$P(\partial_X, \partial_Y) = P(\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}, \partial_{y_1}, \partial_{y_2}, \ldots, \partial_{y_n}). \tag{3}$$

(We write ∂_{x_i} as shorthand for ∂/∂_{x_i}.) With $\boldsymbol{\alpha}$ as above, define the ideal

$$\mathcal{I}_{L[\boldsymbol{\alpha}]} = \left\{ P(X_n, Y_n) \in \mathbb{C}[X_n, Y_n] : P(\partial_X, \partial_Y)\Delta_{L[\boldsymbol{\alpha}]} = 0 \right\} \tag{4}$$

and write $\mathbb{C}[X_n, Y_n]_{\mathcal{I}_{L[\boldsymbol{\alpha}]}}$ for the quotient ring $\mathbb{C}[X_n, Y_n]/\mathcal{I}_{L[\boldsymbol{\alpha}]}$.

These quotients, known as *Garsia-Haiman modules*, were introduced by A. Garsia and M. Haiman in [9]. A good overview of the subject can be found in [12]. A. Garsia and M. Haiman introduced modules of this type to study the q, t-Kostka

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coefficients. This paper will henceforth concern itself only with Garsia-Haiman modules arising from *hollow lattice diagrams*. Roughly, a hollow lattice diagram is a subset of a hook shape obtained by removing a (perhaps trivial) contiguous region of cells from each of the arm and leg of the hook (see Figure 1). More precisely we parametrize hollow lattice diagrams L_γ by sequences of three pairs $\gamma = (m, k, p)$, with $m = (m_1, m_2) \in \mathbb{Z}^2_{\geq 1}, k = (k_1, k_2) \in \mathbb{Z}^2_{\geq 1}$ and $p = (p_1, p_2) \in \mathbb{N}^2$, by setting $L_\gamma = L[\boldsymbol{\alpha}]$ where

$$\begin{aligned}\boldsymbol{\alpha} = (&(0, m_2 + k_2 + p_2 - 1), (0, m_2 + k_2 + p_2 - 2), \ldots, (0, m_2 + k_2),\\ &(0, m_2 + k_2 - 1), (0, m_2 - 1), \ldots, (0, 1), (0, 0), (1, 0), (2, 0),\\ &\ldots, (m_1 - 1, 0), (m_1 + k_1 - 1, 0), \ldots, (m_1 + k_1 + p_1 - 1, 0)) .\end{aligned}$$

(We also allow $k_i = 0$ if $p_i = 0$.) Unless otherwise noted, the number of cells in L_γ (namely, $m_1 + p_1 + m_2 + p_2 + 1$) will be denoted by n.

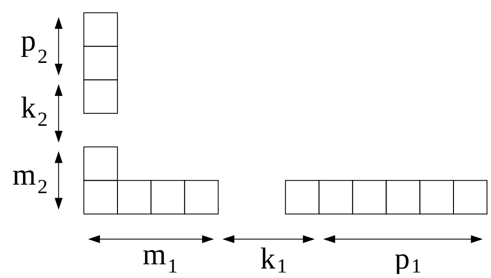

FIGURE 1. Hollow lattice diagram L_γ with $m = (4, 2)$, $k = (3, 2)$, and $p = (5, 2)$.

Abusing notation slightly, we write $\mathcal{I}_\gamma$ for $\mathcal{I}_{L_\gamma}$ and Δ_γ for Δ_{L_γ}. Our goal is to consider the combinatorics of the hollow Garsia-Haiman space $\mathbb{C}[X_n, Y_n]_{\mathcal{I}_\gamma}$. As suggested by the previous terminology, the rings $\mathbb{C}[X_n, Y_n]_{\mathcal{I}_\gamma}$ carry S_n-representations: The symmetric group S_n has a natural *diagonal action* on $\mathbb{C}[X_n, Y_n]$ given by

$$\sigma P(x_1, \ldots, x_n, y_1, \ldots, y_n) = P(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, y_{\sigma(1)}, \ldots, y_{\sigma(n)}). \tag{5}$$

This action passes through to an action on each $\mathbb{C}[X_n, Y_n]_{\mathcal{I}_\gamma}$.

Let R be any S_n-module realized as a polynomial ring over X_n and Y_n (such as $\mathbb{C}[X_n, Y_n]_{\mathcal{I}_\gamma}$) and let $R_{r,s}$ denote the subspace of R containing elements of total degree r in X_n and total degree s in Y_n. We can decompose each $R_{r,s}$ as $R_{r,s} = \oplus_{r,s} c_{r,s} S^\lambda$ where each S^λ is an irreducible S_n-module (i.e., Specht module). Denote the Schur functions by s_λ. The *(bi-graded) character*, *Frobenius series* and *Hilbert series* are then, respectively, given by

$$\begin{aligned}\operatorname{ch}(R) &= \sum_{r,s} \left(\sum_{\lambda \vdash n} c_{r,s} \chi^\lambda \right) t^r q^s,\\ \mathcal{F}\operatorname{ch}(R) &= \sum_{r,s} \left(\sum_{\lambda \vdash n} c_{r,s} s_\lambda \right) t^r q^s, \\ \mathcal{H}(R) &= \sum_{r,s} \dim(R_{r,s})\ t^r q^s.\end{aligned} \tag{6}$$

Here χ^λ denotes the character of S^λ and $\lambda \vdash n$ signifies that λ is a partition of n. The Frobenius series is the image of the graded character under the Frobenius map which sends χ^λ to s_λ. Note that the Hilbert series can be recovered from the Frobenius series by formally replacing each s_λ by the dimension of S^λ.

By constructing an appropriate basis we will prove the following theorem. (The definition of a standard tableau will be given in Section 2; the cocharge statistics $|X(\mathrm{rs}(C_\gamma(T)))|$ and $|Y(\mathrm{rs}(C_\gamma(T)))|$ are defined in Section 6.)

Theorem 1. *Let $\mathbb{C}[X_n, Y_n]_{\mathcal{I}_\gamma}$ denote a hollow Garsia-Haiman module parametrized by $\gamma = (m, p, k)$. The graded character $\mathrm{ch}(\mathbb{C}[X_n, Y_n]_{\mathcal{I}_\gamma})$ of $\mathbb{C}[X_n, Y_n]_{\mathcal{I}_\gamma}$ is given by*

$$\begin{bmatrix} p_1 + k_1 \\ p_1 + 1 \end{bmatrix}_t \begin{bmatrix} p_2 + k_2 \\ p_2 + 1 \end{bmatrix}_q \sum_{\lambda \vdash n} \chi^\lambda \sum_{T \in SYT(\lambda)} t^{|X(\mathrm{rs}(C_\gamma(T)))|} \; q^{|Y(\mathrm{rs}(C_\gamma(T)))|}, \tag{7}$$

where $SYT(\lambda)$ denotes the collection of standard tableau of shape λ.

The modified Macdonald polynomials are a family of symmetric functions over the field of Laurent polynomials in two variables that specialize to many important classical symmetric functions. Parametrized by partitions, they are given plethystically by

$$\tilde{H}_\mu(z; q, t) = J_\mu \left[\frac{Z}{1 - t^{-1}}; q, t^{-1} \right] t^{n(\mu)} = \sum_\lambda \; \tilde{K}_{\lambda,\mu}(q, t) \; s_\lambda(z). \tag{8}$$

An argument analogous to the one used to prove Theorem 2.1 in [6] can be used to deduce the following corollary from Theorem 1.

Corollary 2. *The graded Frobenius characteristic of the hollow Garsia-Haiman module $\mathbb{C}[X_n, Y_n]_{\mathcal{I}_\gamma}$ is given by the polynomial*

$$\mathcal{F} \; \mathrm{ch}(\mathbb{C}[X_n, Y_n]_{\mathcal{I}_\gamma}) = \Xi_\lambda(t) \; \Xi_\theta(q) \; \tilde{H}_{(m_2 + p_2 + 1, 1^{m_1 + p_1})}(z; q, t) \tag{9}$$

where $\lambda = (k_1^{p_1+1})$, $\theta = (k_2^{p_2+1})$ and $\Xi_\nu(t)$ gives the Hilbert polynomial of the graded vector space of skew Schur functions $s_{\nu/\mu}$ as μ varies in ν.

To prove Theorem 1, we will define a sequence of ideals

$$\mathcal{G}_\gamma(X_n, Y_n) \subset \mathcal{H}_\gamma(X_n, Y_n) \subset \mathcal{J}_\gamma(X_n, Y_n) \subset \mathcal{K}_\gamma(X_n, Y_n) \subset \mathcal{I}_\gamma(X_n, Y_n). \tag{10}$$

For each ideal $\mathcal{E} = \mathcal{E}(X_n, Y_n)$ in (10), we will define appropriate generators for $\mathcal{E}$, construct a base for the corresponding quotient space $\mathbb{C}[X_n, Y_n]_{\mathcal{E}} = \mathbb{C}[X_n, Y_n]/\mathcal{E}$, and compute the corresponding Hilbert series. To complete the proof, we will use a correspondence between our basis elements for $\mathbb{C}[X_n, Y_n]_{\mathcal{I}_\gamma}$ and irreducible characters of S_n. Sections 2, 3 and 4 will introduce the necessary background and notation on tableaux, cocharge tableaux and symmetric polynomials, respectively. The following four sections consider the situations corresponding to $\mathcal{G}_\gamma$, $\mathcal{H}_\gamma$, both $\mathcal{J}_\gamma$ and $\mathcal{K}_\gamma$, and both $\mathcal{J}_\gamma$ and $\mathcal{I}_\gamma$, respectively.

We note here that Garsia-Haiman modules corresponding to specific classes of lattice diagrams have been studied elsewhere. *Periodic Garsia-Haiman modules* were considered by the first author [3] and (in one variable) by H. Morita and H.-F. Yamada [15], R. Stanley [17] and J. Stembridge [18]. *Dense Garsia-Haiman modules* were investigated by the first author in [4]. The Garsia-Haiman modules corresponding to the degenerate hollow lattice diagrams of one-row skew shapes or one-column skew shapes were studied by F. Bergeron, A. Garsia and G. Tesler in [6].

Finally, it should be noted that a conjecture has been announced by M. Haiman, J. Haglund, N. Loehr, J. Remmel, and A. Ulyanov (see [11]) for a combinatorial formula for the character of the coinvariants of the symmetric group.

Remark 3. There is a wide variety of indexing and notational conventions among papers in this field. Most obviously, when the primary lattice diagrams under consideration are partitions, the correspondence between $\mathbb{N}^2$ and first quadrant lattice points usually has the first index giving the y-coordinate. Also note that we here write (m, k, p) for the parametrizing tuple $[[1^{m_1}, k_1, 1^{p_1}], [1^{m_2}, k_2, 1^{p_2}]]$ of [4].

2. Tableaux

A partition $\mu = (\mu_1, \mu_2, \ldots, \mu_j, \ldots)$ is a (possibly infinite) sequence of weakly decreasing integers with $j \geq 1$ nonzero terms. We will not distinguish between partitions with the same collection of nonzero terms. Write $|\mu| = \mu_1 + \mu_2 + \cdots + \mu_j$ for the sum of the parts. If $|\mu| = n$, then we say that μ is a *partition* of n and write $\mu \vdash n$. The length, j, is denoted $\ell(\mu)$. If i appears m_i times in μ for each i, then the tuple $(1^{m_1}, 2^{m_2}, \ldots)$ is called the *type* of μ. μ^t will denote the *transpose* of μ.

Let $\mu, \lambda \vdash n$. We use "$\leq_{\text{lex}}$" on partitions to denote the lexicographic order. A (French-style) *Ferrers diagram of shape* μ is a collection of left-justified unit squares ("cells") in the first quadrant with μ_i squares in the ith row from the bottom. The *shape* of a Ferrers diagram D, $\text{sh}(D)$, is the partition obtained by listing the row lengths of D. The notation $\text{dg}(\mu)$ will be used to denote the canonical Ferrers diagram of shape μ.

A Σ-*filling* f is a map $f : \text{dg}(\mu) \longrightarrow \Sigma$ from the cells of a Ferrers diagram to some totally ordered alphabet Σ. We will consider fillings with three different alphabets:

(1) $\mathcal{A}' = \{(a, b) : a, b \in \mathbb{N}\}$ ordered by $(a_1, b_1) <_{\mathcal{A}'} (a_2, b_2)$ whenever
   (a) $a_1 - b_1 < a_2 - b_2$; or
   (b) $a_1 - b_1 = a_2 - b_2$ and $a_1 < a_2$.

   Geometrically, the order $<_{\mathcal{A}'}$ can be visualized as listing the points in the first quadrant by reading down lines $y = x + c$ from left to right with successively smaller values of c.
(2) $\mathcal{A} = \{(a, b) \in \mathcal{A}' : a = 0 \text{ or } b = 0\}$ with the order $<_{\mathcal{A}}$ induced by $<_{\mathcal{A}'}$. Note that the elements of $\mathcal{A}$ index cells that can appear in a hollow lattice diagram. For brevity in formulas, we sometimes write $\underline{a}$ for $(a, 0)$ and $\overline{b}$ for $(0, b)$. The notation is meant to evoke positive and negative numbers, respectively, as this interpretation of the elements of $\mathcal{A}$ is consistent with $<_{\mathcal{A}}$.
(3) $\mathbb{N}$ ordered by $0 < 1 < 2 < 3 < \cdots$.

The picture (or pair $(f, \text{dg}(\mu))$) obtained by placing elements of Σ in the cells of a Ferrers diagram of shape μ according to f is a Σ-*filled diagram.* When Σ is clear (or unimportant), we simply refer to *filled diagrams.* For a filled diagram $U = (f, \text{dg}(\mu))$, we use the shorthand $g(U)$ for the new filled diagram $(g \circ f, \text{dg}(\mu))$.

A filled diagram is *injective* if the map f is an injective map. When $\Sigma = \mathbb{Z}_{\geq 1} \subset \mathbb{N}$, we refer to a Σ-filled diagram as a *tableau.* A filled diagram of shape μ is said to be *column strict* if the entries increase weakly from left to right in each row and increase strictly in each column from bottom to top. We will denote the collection of column-strict tableaux for a given alphabet Σ by $\mathcal{CS}_\Sigma$ (and use $\mathcal{CS}_{n,\Sigma}$ if we want to specify the number of boxes). An injective column-strict tableau with distinct

entries $\{1, \ldots, n\}$ for some n is often referred to as a *standard* tableau. Let SYT, $SYT(\lambda)$, and SYT_n denote the collections of standard tableaux, standard tableaux of shape λ, and standard tableaux with n cells, respectively. In the context of filled diagrams, T will be reserved for a standard tableau; V for a column-strict tableau. Finally, for any filled diagram U, we define U^t to be the tableau obtained by reflecting U along the line $y = x$. Figure 2 illustrates an $\mathcal{A}$-filled diagram U of shape $\mathrm{sh}(U) = (3, 2)$ along with its transpose U^t. Note that in this case, $U^t \in \mathcal{CS}_{\mathcal{A}}$.

$$U = \begin{array}{|c|c|c}\hline (0,2) & (0,1) & \\ \hline (0,2) & (0,0) & \multicolumn{1}{c|}{(3,0)} \\ \hline \end{array} \qquad\qquad U^t = \begin{array}{|c|c|}\hline (3,0) & \multicolumn{1}{c}{} \\ \hline (0,0) & (0,1) \\ \hline (0,2) & (0,2) \\ \hline \end{array}$$

FIGURE 2. An $\mathcal{A}$-filled diagram and its transpose.

Let I denote an injective tableau of shape $\mu = (\mu_1, \ldots, \mu_j)$, R_i $(1 \le i \le j)$ denote the collection of integers in the i^{th} row of I and D_i $(1 \le i \le \mu_1)$ denote the collection of integers in the i^{th} column of I. Set

$$R(I) = S_{R_1} \times S_{R_2} \times \cdots \times S_{R_j} \tag{11}$$

and

$$D(I) = S_{D_1} \times S_{D_2} \times \cdots \times S_{D_{\mu_1}}, \tag{12}$$

where S_{R_i} and S_{D_i} denote the symmetric group on the collections of elements R_i and D_i, respectively. Define, in the group algebra $\mathbb{C}[S_n]$,

$$P(I) = \sum_{\sigma \in R(I)} \sigma \quad \text{and} \quad N(I) = \sum_{\sigma \in D(I)} \mathrm{sgn}(\sigma)\, \sigma. \tag{13}$$

A *bitableau* is a pair (S, U) of filled diagrams of the same shape where S is $\mathbb{N}$-filled and U is $\mathcal{A}'$-filled. A *standard bitableau* satisfies the additional stipulations that S is a standard tableau and $U \in \mathcal{CS}_{\mathcal{A}'}$. The set of all standard bitableaux (S, U) on n boxes will be denoted Θ_n if we restrict to $U \in \mathcal{CS}_{n,\mathcal{A}}$ and Θ'_n if we do not place this restriction.

Given any Σ-filled diagram $U = (f, \mathrm{dg}(\mu))$, define the *standardization* $\mathrm{std}(U) = (\xi_f, \mathrm{dg}(\mu))$ by setting ξ_f to be the unique standard filling such that: For cells $c, d \in \mathrm{dg}(\mu)$,

1. $f(c) \le_\Sigma f(d)$ implies $\xi_f(c) < \xi_f(d)$.
2. If $f(c) = f(d)$ and either
   (a) c is north of d, or
   (b) c is in the same row as d but west,
   then $\xi_f(c) < \xi_f(d)$.

For U a Σ-filled diagram and T a standard tableau, we denote the entry in U in the cell corresponding to i in T by u_i^T. Two special cases that arise frequently are when $T = \mathrm{std}(U)$ and when (T, U) is a bitableau. In the former case, we abbreviate u_i^T by u_i.

Let S be an injective $\mathbb{N}$-filled tableau. Again writing $z_j^{u_i^S}$ for $x_j^{u_{i,1}^S} y_j^{u_{i,2}^S}$, for a bitableau (S, U) we set the *bideterminant* $[S, U]_{\text{det}}$ to be

$$[S,U]_{\text{det}} = N(S)\ z_1^{u_1^S} z_2^{u_2^S} \cdots z_n^{u_n^S} = \sum_{\sigma \in D(S)} \operatorname{sgn}(\sigma) z_{\sigma(1)}^{u_1^S} z_{\sigma(2)}^{u_2^S} \cdots z_{\sigma(n)}^{u_n^S}. \tag{14}$$

Similarly, the corresponding *bipermanent* $[S, U]_{\text{per}}$ is given by

$$[S,U]_{\text{per}} = P(S)\ z_1^{u_1^S} z_2^{u_2^S} \cdots z_n^{u_n^S} = \sum_{\sigma \in R(S)} z_{\sigma(1)}^{u_1^S} z_{\sigma(2)}^{u_2^S} \cdots z_{\sigma(n)}^{u_n^S}. \tag{15}$$

The following theorem is a special case of [10, Theorem 8] (also cf. [4, 7]). We suggest the reader work out some examples from the case $n = 3$ by hand.

Lemma 4. *The collections*

$$\mathcal{BP} = \{[T,V]_{\text{per}} : (T,V) \in \Theta_n'\} \ \ \textit{and} \tag{16}$$

$$\mathcal{BD} = \{[T,V]_{\text{det}} : (T,V) \in \Theta_n'\} \tag{17}$$

are infinite bases for $\mathbb{C}[X_n, Y_n]$.

For a Σ-filled diagram U, the *row sequence* $\operatorname{rs}(U)$ is the sequence obtained by listing the entries of U in each row from left to right, starting with the bottom row. The *column sequence* $\operatorname{cs}(U)$ is found by listing the entries of U from bottom to top in each column, starting with the leftmost column. Finally, the *content* $\kappa(U)$ is a rearrangement of the row sequence $\operatorname{rs}(U)$ of U into nondecreasing order with respect to $<_\Sigma$.

Example 5. For U as in Figure 2, we have

$$\begin{aligned} \operatorname{rs}(U) &= ((0,2),(0,0),(3,0),(0,2),(0,1)) = \left(\overline{2},\underline{0},\underline{3},\overline{2},\overline{1}\right), \\ \operatorname{cs}(U) &= ((0,2),(0,2),(0,0),(0,1),(3,0)) = \left(\overline{2},\overline{2},\underline{0},\overline{1},\underline{3}\right), \text{ and} \\ \kappa(U) &= ((0,2),(0,2),(0,1),(0,0),(3,0)) = \left(\overline{2},\overline{2},\overline{1},\underline{0},\underline{3}\right). \end{aligned}$$

There are two orderings of bitableaux that are particularly important when considering elements of $\mathcal{BD}$ or $\mathcal{BP}$. Let $>_{\text{lex}(\mathcal{A}')}$ denote the lexicographic order with respect to $>_{\mathcal{A}'}$. For tableau, S_1, U_1, S_2 and U_2, we will say that

$$(S_1, U_1) <_{\text{det}} (S_2, U_2) \tag{18}$$

whenever

(1) $\operatorname{sh}(S_1^t) <_{\text{lex}} \operatorname{sh}(S_2^t)$;
(2) If $\operatorname{sh}(S_1) = \operatorname{sh}(S_2)$ then $\kappa(U_1) >_{\text{lex}(\mathcal{A}')} \kappa(U_2)$;
(3) If $\operatorname{sh}(S_1) = \operatorname{sh}(S_2)$ and $\kappa(U_1) = \kappa(U_2)$ then

$$\operatorname{cs}(S_1)\operatorname{cs}(U_1) >_{\text{lex}(\mathcal{A}')} \operatorname{cs}(S_2)\operatorname{cs}(U_2), \tag{19}$$

where $\operatorname{cs}(S_i)\operatorname{cs}(U_i)$ is the concatenation of $\operatorname{cs}(S_i)$ and $\operatorname{cs}(U_i)$ for $i = 1, 2$.

Example 6. Let

$$(S_1,U_1) = \left(\begin{array}{|c|c|c} \hline 5 & 7 & \multicolumn{1}{|c}{} \\ \hline 3 & 4 & \multicolumn{1}{|c|}{8} \\ \hline 1 & 2 & \multicolumn{1}{|c|}{6} \\ \hline \end{array}, \begin{array}{|c|c|c} \hline \underline{\underline{0}} & \underline{1} & \multicolumn{1}{|c}{} \\ \hline \overline{\overline{1}} & \underline{\underline{0}} & \multicolumn{1}{|c|}{\underline{\underline{1}}} \\ \hline \overline{2} & \overline{\overline{1}} & \multicolumn{1}{|c|}{\overline{\overline{1}}} \\ \hline \end{array} \right) \text{ and } (S_2,U_2) = \left(\begin{array}{|c|c|c} \hline 5 & 7 & \multicolumn{1}{|c}{} \\ \hline 3 & 4 & \multicolumn{1}{|c|}{8} \\ \hline 1 & 2 & \multicolumn{1}{|c|}{6} \\ \hline \end{array}, \begin{array}{|c|c|c} \hline \underline{\underline{1}} & \underline{1} & \multicolumn{1}{|c}{} \\ \hline \overline{\overline{1}} & \underline{\underline{0}} & \multicolumn{1}{|c|}{\underline{\underline{1}}} \\ \hline \overline{2} & \overline{\overline{2}} & \multicolumn{1}{|c|}{\overline{\overline{1}}} \\ \hline \end{array} \right).$$

Certainly $\operatorname{sh}(S_1^t) = \operatorname{sh}(S_2^t)$. However

$$\kappa(U_1) = (\overline{2},\overline{1},\overline{1},\overline{1},\underline{0},\underline{0},\underline{1},\underline{1}) >_{\text{lex}(\mathcal{A})} \kappa(U_2) = (\overline{2},\overline{2},\overline{1},\overline{1},\underline{0},\underline{1},\underline{1},\underline{1}). \tag{20}$$

So $(S_1, U_1) <_{\text{det}} (S_2, U_2)$.

Similarly, we will say that

$$(S_1, U_1) <_{\text{per}} (S_2, U_2) \tag{21}$$

whenever

(1) $\text{sh}(S_1) <_{\text{lex}(\mathcal{A})} \text{sh}(S_2)$;
(2) If $\text{sh}(S_1) = \text{sh}(S_2)$ then $\kappa(U_1) <_{\text{lex}(\mathcal{A}')} \kappa(U_2)$;
(3) If $\text{sh}(S_1) = \text{sh}(S_2)$ and $\kappa(U_1) = \kappa(U_2)$ then

$$\text{rs}(S_1)\,\text{rs}(U_1) >_{\text{lex}(\mathcal{A}')} \text{rs}(S_2)\,\text{rs}(U_2), \tag{22}$$

where $\text{rs}(S_i)\,\text{rs}(U_i)$ is the concatenation of $\text{rs}(S_i)$ and $\text{rs}(U_i)$ for $i = 1, 2$.

Theorem 7 ([4, 7, 10])**.** *Let $[S, U]_{\text{det}}$ be a bitableau on n boxes with U $\mathcal{A}'$-filled such that either S is not standard or U is not column-strict. Then we can write*

$$[S, U]_{\text{det}} = \sum_i d_i [T_i, V_i]_{\text{det}} \tag{23}$$

where, for each i, it is true that $d_i \in \mathbb{Z}$, $(T_i, V_i) \in \Theta'_n$,

$$(T_i, V_i) >_{\text{det}} (S, U),$$

$\kappa(T_i) = \kappa(S)$ and $\kappa(V_i) = \kappa(U)$. The above statements hold, mutatis mutandi*, for bipermanents and the order $>_{\text{per}}$.*

Example 8.

$$\left[\begin{array}{|c|c|} \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 2 & 1 \\ \hline \end{array}, \begin{array}{|c|c|} \cline{1-1} \underline{\underline{2}} & \multicolumn{1}{c}{} \\ \hline \overline{\overline{1}} & \underline{1} \\ \hline \end{array}\right]_{\det} = (\varepsilon - (2,3))x_1 y_2 x_3^2 = x_1 y_2 x_3^2 - x_1 y_3 x_2^2$$

$$= \left[\begin{array}{|c|c|} \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 1 & 2 \\ \hline \end{array}, \begin{array}{|c|c|} \cline{1-1} \underline{\underline{2}} & \multicolumn{1}{c}{} \\ \hline \overline{\overline{1}} & \underline{1} \\ \hline \end{array}\right]_{\det} - \left[\begin{array}{|c|c|} \cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 3 \\ \hline \end{array}, \begin{array}{|c|c|} \cline{1-1} \underline{\underline{2}} & \multicolumn{1}{c}{} \\ \hline \overline{\overline{1}} & \underline{1} \\ \hline \end{array}\right]_{\det} - \left[\begin{array}{|c|} \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array}, \begin{array}{|c|} \hline \underline{\underline{2}} \\ \hline \underline{1} \\ \hline \overline{\overline{1}} \\ \hline \end{array}\right]_{\det}.$$

3. Cocharge Tableaux

Given a hollow lattice diagram L_γ, to any standard tableau $T = (f, \text{dg}(\mu))$, we define the *cocharge diagram*, $C_\gamma(T) = (h_\gamma \circ \pi \circ f, \text{dg}(\mu))$. Here, π is the usual "cocharge" map defined recursively by

$$\pi(i) = \begin{cases} 0, & \text{if } i = 1, \\ \pi(i-1), & \text{if } i > 1 \text{ occurs weakly southeast of } i-1 \text{ in T}, \\ \pi(i-1) + 1, & \text{if } i > 1 \text{ occurs weakly northwest of } i-1 \text{ in T}. \end{cases} \tag{24}$$

The map π is well-defined on standard diagrams. We then define h_γ as the unique order- and cover-preserving map from $\mathbb{N}$ to $\mathcal{A}$ that sends $f^{-1}(m_2 + p_2 + 1)$ to $(0, 0)$.

Figure 3 gives an example cocharge diagram that corresponds to any $\gamma = (m, k, p)$ describing a lattice diagram with 9 boxes such that $m_2 + p_2 + 1$ is equal to 4 or 5.

Lemma 9. *Fix n and γ. Define $\mathcal{CO}_{n,\gamma} = \{C_\gamma(T) : T \in SYT_n\}$. There is a bijection between elements $U \in \mathcal{CS}_{n,\mathcal{A}}$ with $u_{m_2+p_2+1} = \underline{0}$ and pairs $(C, \boldsymbol{\alpha})$ with $C \in \mathcal{CO}_{n,\gamma}$ and $\boldsymbol{\alpha} \in \mathcal{A}^n$ such that $\alpha_i \leq_{\mathcal{A}} \alpha_{i+1}$ for $1 \leq i \leq n-1$ and $\alpha_{m_2+p_2+1} = (0, 0)$.*

(0,2)	(4,0)		
(0,5)	(1,0)	(4,0)	
(0,7)	(0,7)	(0,0)	(3,0)

4	8		
3	6	9	
1	2	5	7

2	4		
1	3	4	
0	0	2	3

(0,0)	(2,0)		
(0,1)	(1,0)	(2,0)	
(0,2)	(0,2)	(0,0)	(1,0)

FIGURE 3. A diagram U, its standardization $\mathrm{std}(U)$, $\pi(\mathrm{std}(U))$ and $C_\gamma(\mathrm{std}(U))$.

For an $\mathcal{A}$-filled diagram as in Figure 3, the sequence $\boldsymbol{\alpha}$ of Lemma 9 is

$$\boldsymbol{\alpha} = ((0,5),(0,5),(0,4),(0,2),(0,0),(0,0),(2,0),(2,0),(2,0)).$$

Proof. Let $T = \mathrm{std}(U)$ and $C = C_\gamma(T)$. Map U to $(C, \boldsymbol{\alpha})$ with $\alpha_i = u_i - c_i^T$ for each $1 \leq i \leq n$. By the definition of h_γ, $c^T_{m_2+p_2+1} = \underline{0}$. Combined with our requirement for $u_{m_2+p_2+1}$, it follows that $\alpha_{m_2+p_2+1} = \underline{0}$. By the definitions of h_γ, π and standardization, the sequences $c_1^T, c_2^T, \ldots$ and $u_1, u_2, \ldots$ are both weakly increasing. That the α_i are weakly increasing then follows from the additional fact that U is column-strict. $\square$

4. Some Operations by Symmetric Polynomials

We now review some important definitions and results with respect to symmetric polynomials. A standard reference for this material is [14]. We introduce the following families of symmetric polynomials. We use the convention that tuples of elements of $\mathcal{A}'$ or $\mathcal{A}$ are written in boldface. In these definitions, let $\lambda = (\lambda_1, \ldots, \lambda_j, \ldots, \lambda_n)$ be a partition with $j \leq n$ indexing the last nonzero part.

(1) Define the *monomial symmetric function* as

$$m_\lambda(X_n) = \sum_{\nu=(\nu_1,\nu_2,\ldots,\nu_n)} x_1^{\nu_1}\, x_2^{\nu_2} \cdots x_n^{\nu_n}, \tag{25}$$

where the sum is over all distinct permutations ν of λ.

(2) For a sequence $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n) \in (\mathcal{A}')^n$, the *MacMahon monomial symmetric function*

$$m_{\boldsymbol{\beta}}(X_n, Y_n) = \sum_{\boldsymbol{\delta}=(\delta_1,\delta_2,\ldots,\delta_n)} z_1^{\delta_1} z_2^{\delta_2} \cdots z_n^{\delta_n}, \tag{26}$$

where the sum is over all distinct permutations $\boldsymbol{\delta}$ of $\boldsymbol{\beta}$.

(3) For a positive integer r, the *elementary symmetric function*

$$e_r(X_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq n} x_{i_1} x_{i_2} \cdots x_{i_r}. \tag{27}$$

Set $e_0 = 1$ and $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_j}$.

(4) For a positive integer r, the *complete (or homogeneous) symmetric function*

$$h_r(X_n) = \sum_{|\lambda|=r} m_\lambda(X_n). \tag{28}$$

Set $h_0 = 1$ and $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_j}$ (we extend this definition in the obvious way to the case where λ is a j-tuple of nonnegative integers; i.e., not necessarily nonincreasing).

Lemma 10 ([3]; Theorem 5.4, Corollary 5.5)**.** *Suppose $g : \mathcal{CS}_{n,\mathcal{A}'} \to \mathcal{CS}_{n,\mathcal{A}'}$ such that for all $W \in \mathcal{CS}_{\mathcal{A}'}$, W and $g(W)$ have the same standardization. Fix $T \in SYT_n$, $V \in \mathcal{CS}_{\mathcal{A}'}$ and write $U = g(V)$. For each $1 \le i \le n$, set $\beta_i = v_i - u_i$. Write $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$. Suppose that $\boldsymbol{\beta} \in \mathcal{A}^n$ and $\beta_i \le_{\mathcal{A}} \beta_{i+1}$ for $1 \le i \le n-1$. Then*

$$m_{\boldsymbol{\beta}}(X_n, Y_n)[T,U]_{\text{per}} = c_{T,V}[T,V]_{\text{per}} + \sum_{\substack{(S,W) >_{\text{per}} (T,V) \\ S \in SYT_n,\, W \in \mathcal{CS}_{n,\mathcal{A}}}} c_{S,W}[S,W]_{\text{per}} + \sum_{\tilde{S} \in SYT_n} c_{\tilde{S},\tilde{W}}[\tilde{S},\tilde{W}]_{\text{per}}, \tag{29}$$

where $c_{T,V} \neq 0$ and $\tilde{W}$ is $\mathcal{A}'$-filled with at least one entry not in $\mathcal{A}$.

Example 11. For $\boldsymbol{\beta} = (\bar{2}, \underline{0}, \underline{3})$, $U = \begin{array}{ll} \boxed{\underline{1}} & \\ \boxed{\underline{0}} & \boxed{\underline{0}} \end{array}$, $T = \begin{array}{ll} \boxed{3} & \\ \boxed{1} & \boxed{2} \end{array}$ and $V = \begin{array}{ll} \boxed{\underline{4}} & \\ \boxed{\bar{2}} & \boxed{\underline{0}} \end{array}$, we have

$$m_{\boldsymbol{\beta}}(X_3, Y_3)\,[T,U]_{\text{per}} = 2[T,V]_{\text{per}} + 2\left[T, \begin{array}{ll} \boxed{\underline{1}} & \\ \boxed{\bar{2}} & \boxed{\underline{3}} \end{array}\right]_{\text{per}} + 2\left[T, \begin{array}{ll} \boxed{a} & \\ \boxed{\underline{0}} & \boxed{\underline{3}} \end{array}\right]_{\text{per}}, \tag{30}$$

where $a = (1,2)$. We see that the second bipermanent is, in fact, larger than (T,V) in the order $>_{\text{per}}$ as the content $(\bar{2}, \underline{1}, \underline{3})$ is greater than that of V in the lexicographic order with respect to $\mathcal{A}$.

We also have the following Lemma (cf. [3], Theorem 5.2).

Lemma 12. *For $\boldsymbol{\beta} \in (\mathcal{A}')^n$ and a lattice diagram $L[\boldsymbol{\alpha}]$,*

$$m_{\boldsymbol{\beta}}(\partial_X, \partial_Y)\ \Delta_{L[\boldsymbol{\alpha}]} = \sum_{\boldsymbol{\delta}} c_{\boldsymbol{\delta}}\, \Delta_{L[\boldsymbol{\alpha}-\boldsymbol{\delta}]} \tag{31}$$

for some constants $c_{\boldsymbol{\delta}} \in \mathbb{N}$. Here, the sum is over all distinct permutations $\boldsymbol{\delta}$ of $\boldsymbol{\beta}$. We use the convention that $c_{\boldsymbol{\delta}} = 0$ if $\alpha_i - \delta_i \notin \mathcal{A}$ for some $1 \le i \le n$.

Proof. First note that $m_{\boldsymbol{\beta}}(\partial_X, \partial_Y)$ can be written as

$$m_{\boldsymbol{\beta}}(\partial_X, \partial_Y) = \sum_{\boldsymbol{\delta}} \prod_{i=1}^{n} \partial_{x_i}^{\delta_{i,1}} \partial_{y_i}^{\delta_{i,2}} = K \sum_{\nu \in S_n} \prod_{i=1}^{n} \partial_{x_i}^{\beta_{\nu(i),1}} \partial_{y_i}^{\beta_{\nu(i),2}}$$

for some constant $K \in \mathbb{Q}$ dependent on the extent to which factors in $\boldsymbol{\beta}$ are repeated. Then,

$$\begin{aligned} m_{\boldsymbol{\beta}}(\partial_X, \partial_Y)\ \Delta_{L[\boldsymbol{\alpha}]} &= \sum_{\nu \in S_n} K \prod_{i=1}^{n} \partial_{z_i}^{\beta_{\nu(i)}} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) z_1^{\alpha_{\sigma^{-1}(1)}} \cdots z_n^{\alpha_{\sigma^{-1}(n)}} \\ &= \sum_{\nu \in S_n} c_\nu \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} z_i^{\alpha_{\sigma^{-1}(i)} - \beta_{\nu(i)}} \\ &= \sum_{\phi \in S_n} c_\phi \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} z_i^{\alpha_{\sigma^{-1}(i)} - \beta_{\sigma^{-1}\phi(i)}} \\ &= \sum_{\boldsymbol{\delta}} c_{\boldsymbol{\delta}}\, \Delta_{L[\boldsymbol{\alpha}-\boldsymbol{\delta}]} \end{aligned} \tag{32}$$

for some constants $c_{\boldsymbol{\delta}}$. In the above, we consider the coefficient c_ν to be zero if the exponent of any of the z_i's is not in $\mathcal{A}$. In addition, we have $\nu = \sigma^{-1}\phi$ and let $\boldsymbol{\delta}$ run over all distinct permutations of $\boldsymbol{\beta}$. $\square$

5. The Ideal $\mathcal{G}_\gamma(X_n, Y_n)$ and the ring $\mathbb{C}[X_n, Y_n]_{\mathcal{G}_\gamma}$.

Let $\mathcal{G}_\gamma(X_n, Y_n)$ be the ideal generated by the monomials in the collections

$$\{x_1y_1, \ldots, x_ny_n\}, \tag{33}$$

$$\left\{\prod_{i\in D} x_i\right\}_{\substack{D\subset\{1,2,\ldots,n\}\\ |D|=m_1+p_1+1}}, \text{ and } \left\{\prod_{h\in E} y_h\right\}_{\substack{E\subset\{1,2,\ldots,n\}\\ |E|=m_2+p_2+1}}. \tag{34}$$

Note that each of the monomials in equations (33) and (34) is in the ideal $\mathcal{I}_\gamma(X_n, Y_n)$. Thus $\mathcal{G}_\gamma(X_n, Y_n) \subset \mathcal{I}_\gamma(X_n, Y_n)$.

For a sequence $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in (\mathcal{A})^n$, we define

$$\begin{aligned} |X(\boldsymbol{\alpha})| &= \alpha_{1,1} + \alpha_{2,1} + \cdots + \alpha_{n,1} \text{ and} \\ |Y(\boldsymbol{\alpha})| &= \alpha_{1,2} + \alpha_{2,2} + \cdots + \alpha_{n,2}. \end{aligned} \tag{35}$$

For an arbitrary bipermanent $b = [T, V]_{\text{per}}$, we write $|X(b)|$ as shorthand for $|X(\text{rs}(V))|$; similarly for $|Y(b)|$. In addition we will write

$$(q)_j = (1-q)(1-q^2)\cdots(1-q^j) \text{ and } (t)_j = (1-t)(1-t^2)\cdots(1-t^j) \tag{36}$$

for the *rising factorial products.* The generating function for the sum

$$\sum_{\boldsymbol{\alpha}} t^{|X(\boldsymbol{\alpha})|}\ q^{|Y(\boldsymbol{\alpha})|}, \tag{37}$$

subject to the constraints that $\boldsymbol{\alpha} \in \mathcal{A}^n$, $\alpha_i \leq_{\mathcal{A}} \alpha_{i+1}$ for $1 \leq i \leq n-1$ and $\alpha_{m_2+p_2+1} = (0,0)$, is given by

$$\sum_{\boldsymbol{\alpha}} t^{|X(\boldsymbol{\alpha})|}\ q^{|Y(\boldsymbol{\alpha})|} = \frac{1}{(t)_{m_1+p_1}} \frac{1}{(q)_{m_2+p_2}}. \tag{38}$$

Define

$$\mathcal{B}_\gamma = \{[T, V]_{\text{per}} :\ T \in SYT_n,\ V \in \mathcal{CO}_{n,\gamma}\}. \tag{39}$$

Theorem 13. *The Hilbert series of* $\mathbb{C}[X_n, Y_n]_{\mathcal{G}_\gamma}$ *is given by*

$$\mathcal{H}(\mathbb{C}[X_n, Y_n]_{\mathcal{G}_\gamma}) = \frac{1}{(t)_{m_1+p_1}} \frac{1}{(q)_{m_2+p_2}} \sum_{b\in\mathcal{B}_\gamma} t^{|X(b)|}\ q^{|Y(b)|}. \tag{40}$$

Proof. Suppose $(T, V) \in \Theta'_n$ and $v_{m_2+p_2+1} \neq (0,0)$. Since $V \in \mathcal{CS}_{\mathcal{A}'}$, either each of the monomials in $[T, V]_{\text{per}}$ has at least $m_1 + p_1 + 1$ distinct x_i's as factors, or each of the monomials has at least $m_2 + p_2 + 1$ distinct y_i's as factors. In either case, $[T, V]_{\text{per}} \in \mathcal{G}_\gamma(X_n, Y_n)$ as can be seen by examining the sets in (34). It follows that in looking for a basis of $\mathbb{C}[X_n, Y_n]_{\mathcal{G}_\gamma}$, we can restrict our attention to those bipermanents for which $v_{m_2+p_2+1} = (0,0)$.

Additionally, if $v_i \notin \mathcal{A}$ for some $1 \leq i \leq n$, then each monomial of $[T, V]_{\text{per}}$ is a multiple of x_jy_j for some j. (Note that j need not equal i as $\text{std}(T)$ need not equal $\text{std}(V)$.) Examination of (33) then shows that $[T, V]_{\text{per}} \in \mathcal{G}_\gamma(X_n, Y_n)$ in this case as well.

On the other hand, it is easily seen that if $v_{m_2+p_2+1} = (0,0)$ and $V \in \mathcal{CS}_{\mathcal{A}}$, then $[T, V]_{\text{per}} \notin \mathcal{G}_\gamma(X_n, Y_n)$. It follows from Lemma 4 that the set

$$\Big\{[T, V]_{\text{per}} : (T, V) \in \Theta_n,\ v_{m_2+p_2+1} = (0,0)\Big\} \tag{41}$$

is a basis for $\mathbb{C}[X_n, Y_n]_{\mathcal{G}_\gamma}$.

The theorem then follows by combining the decomposition of Lemma 9 with the basis of (41) and the generating function of (38). □

Although the above proof constructs a basis of $\mathbb{C}[X_n, Y_n]_{\mathcal{G}_\gamma}$, it will be more useful in the next section to have the basis offered by Theorem 14.

Theorem 14. *The collection*

$$\mathcal{EEB} = \left\{ \left(e_1^{\epsilon_1}(X_n) \cdots e_{m_1+p_1}^{\epsilon_{m_1+p_1}}(X_n)\; e_1^{\beta_1}(Y_n) \cdots e_{m_2+p_2}^{\beta_{m_2+p_2}}(Y_n) \right) b : b \in \mathcal{B}_\gamma \right\}, \tag{42}$$

where the ϵ_i and β_i are allowed to run over all nonnegative integers, is a basis for $\mathbb{C}[X_n, Y_n]_{\mathcal{G}_\gamma}$.

Proof. For this proof, linearly extend the notation $|X(p)|$ to apply to elements $p \in \mathcal{EEB}$.

By the proof of Theorem 13, it suffices to consider a bideterminant $[T, V]_{\text{per}}$ where $(T, V) \in \Theta_n$ and $v_{m_2+p_2+1} = (0, 0)$. Set $C = C_\gamma(\text{std}(V))$. Note that $c_{m_2+p_2+1} = (0, 0)$ by construction.

If $c_i = v_i$ for all $1 \le i \le n$, then $V \in \mathcal{CO}_{n,\gamma}$; hence, by definition, $[T, V]_{\text{per}} \in \mathcal{B}_\gamma$. Assume not. We consider two possibilities.

Suppose there exists such an index greater than $m_2 + p_2 + 1$. Choose i to be the smallest such index. Let U denote the tableau for which $u_j = v_j$ for $1 \le j < i$ and $u_j = v_j - (1, 0)$ for $i \le j \le n$. It follows from the proof of Lemma 9 that $U \in \mathcal{CS}_{n,\mathcal{A}}$. Note that that $(T, U) >_{\text{per}} (T, V)$.

Utilizing the identity $e_{n-i+1} = m_{1^{n-i+1}}$, it follows from Lemma 10 that

$$e_{n-i+1}(X_n)\, [T, U]_{\text{per}} \equiv c_{T,V}\, [T, V]_{\text{per}} + \sum_{\substack{(S,W) >_{\text{per}} (T,V) \\ S \in SYT_n,\ W \in \mathcal{CS}_{n,\mathcal{A}}}} c_{S,W}\, [S, W]_{\text{per}} \mod (\mathcal{G}_\gamma). \tag{43}$$

(Here we have used the fact that the monomials in $e_{n-i+1}(X_n)\ [T, U]_{\text{per}}$ arising in the third term in (29) all have a factor $x_j y_j$ for some j. But, as we see from (33), these monomials are in $\mathcal{G}_\gamma$ by construction.) The only remaining possibility is that there exists such an index i less than $m_2 + p_2 + 1$. Choose i to be the largest such i. Arguing as above, we obtain an equivalent expansion for $e_i(Y_n)$.

Iteration of the above argument on the $[T, U]_{\text{per}}$ and $[S, W]_{\text{per}}$ implies that the collection $\mathcal{EEB}$ spans $\mathbb{C}[X_n, Y_n]_{\mathcal{G}_\gamma}$. (We use the facts that at most $m_1 + p_1$ of the x_i, and $m_2 + p_2$ of the y_i, can appear with positive degree in any of the monomials not in $\mathcal{G}_\gamma$.)

For any $b \in \mathcal{B}_\gamma$, let $\text{pow}(\boldsymbol{\epsilon}, X_n, b)$ denote the X_n-degree of $\prod_{i=1}^{m_1+p_1} e_i^{\epsilon_i}(X_n) b$ and $\text{pow}(\boldsymbol{\beta}, Y_n, b)$ denote the Y_n-degree of $\prod_{i=1}^{m_2+p_2} e_i^{\beta_i}(Y_n) b$. It follows then that

$$\begin{aligned} \sum_{p \in \mathcal{EEB}} t^{|X(p)|}\, q^{|Y(p)|} &= \sum_{\boldsymbol{\epsilon} \in \mathbb{N}^{m_1+p_1}} \sum_{\boldsymbol{\beta} \in \mathbb{N}^{m_2+p_2}} \sum_{b \in \mathcal{B}_\gamma} t^{\text{pow}(\boldsymbol{\epsilon}, X_n, b)} q^{\text{pow}(\boldsymbol{\beta}, Y_n, b)} \\ &= \frac{1}{(t)_{m_1+p_1}} \frac{1}{(q)_{m_2+p_2}} \sum_{b \in \mathcal{B}_\gamma} t^{|X(b)|} q^{|Y(b)|}. \end{aligned} \tag{44}$$

The fact that the collection $\mathcal{EEB}$ spans $\mathbb{C}[X_n, Y_n]_{\mathcal{G}_\gamma}$ and yields the desired Hilbert series (see Theorem 13) implies that $\mathcal{EEB}$ must be a basis for $\mathbb{C}[X_n, Y_n]_{\mathcal{G}_\gamma}$. □

We have the following corollary:

Corollary 15. *The collection*

$$\{e_1(X_n), e_2(X_n), \ldots, e_{m_1+p_1}(X_n), e_1(Y_n), e_2(Y_n), \ldots, e_{m_2+p_2}(Y_n)\} \tag{45}$$

is algebraically independent in the ring $\mathbb{C}[X_n, Y_n]_{\mathcal{G}_\gamma}$.

Proof. Any nontrivial algebraic dependence amongst the elements of (45) would yield a linear dependence amongst the elements of $\mathcal{EEB}$, conflicting with its role in the basis for $\mathbb{C}[X_n, Y_n]_{\mathcal{G}_\gamma}$. $\square$

6. The ideal $\mathcal{H}_\gamma(X_n, Y_n)$ and the ring $\mathbb{C}[X_n, Y_n]_{\mathcal{H}_\gamma}$.

Let $\mathcal{H}_\gamma(X_n, Y_n)$ be the ideal in $\mathbb{C}[X_n, Y_n]$ generated by the collections of monomials in equations (33) and (34) as well as by the elementary symmetric polynomials in the collection

$$\{e_{p_1+2}(X_n), \ldots, e_{p_1+m_1}(X_n), e_{p_2+2}(Y_n), \ldots, e_{p_2+m_2}(Y_n)\}\,. \tag{46}$$

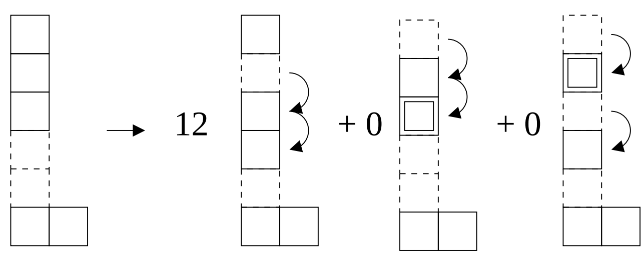

Figure 4. Illustration of the fact that $e_2(\partial_Y)\Delta_{L[(\bar{5},\bar{4},\bar{3},\underline{0},\underline{1})]}$ equals $12\Delta_{L[(\bar{5},\bar{3},\bar{2},\underline{0},\underline{1})]}$.

Consider the action of an elementary symmetric differential operator $e_j(\partial_{Y_n})$, such as is illustrated in Figure 4 for $j = 2$. This operator moves each of j distinct cells down by one place. Any configuration in which two cells end up in the same position or in which a cell moves to a position not indexed by an element of $\mathcal{A}$ contributes zero. (The action of an $e_l(\partial_{X_n})$ is similar.) It follows that for any contributing monomial, the cells contiguous with $(0,0)$ are not moved. But any $e_{p_1+j}(\partial_X)$ for $j > 1$ or $e_{p_2+\ell}(\partial_Y)$ for $\ell > 1$ *must* move one of these fixed cells. So $\mathcal{H}_\gamma(X_n, Y_n) \subset \mathcal{I}_\gamma(X_n, Y_n)$. Note that by construction, $\mathcal{H}_\gamma(X_n, Y_n) \supset \mathcal{G}_\gamma(X_n, Y_n)$. Theorem 14 and Corollary 15 imply the following two corollaries:

Corollary 16. *The set*

$$\left\{ \left(\prod_{i=1}^{p_1+1} e_i^{\epsilon_i}(X_n) \cdot \prod_{j=1}^{p_2+1} e_j^{\epsilon_j}(Y_n) \right) b : b \in \mathcal{B}_\gamma \right\}_{\substack{\boldsymbol{\epsilon} \in \mathbb{N}^{p_1+1} \\ \boldsymbol{\beta} \in \mathbb{N}^{p_2+1}}}, \tag{47}$$

is a basis for $\mathbb{C}[X_n, Y_n]_{\mathcal{H}_\gamma}$.

Corollary 17. *The Hilbert series of* $\mathbb{C}[X_n, Y_n]_{\mathcal{H}_\gamma}$ *is given by*

$$\mathcal{H}(\mathbb{C}[X_n, Y_n]_{\mathcal{H}_\gamma}) = \frac{1}{(t)_{p_1+1}} \frac{1}{(q)_{p_2+1}} \sum_{b \in \mathcal{B}_\gamma} t^{|X(b)|} q^{|Y(b)|}. \tag{48}$$

7. The ideals $\mathcal{J}_\gamma(X_n, Y_n)$ and $\mathcal{K}_\gamma(X_n, Y_n)$

In the previous section, we considered symmetric functions whose corresponding differential operators moved collections of boxes, but for which each cell was only moved one place. Monomial symmetric functions yield operators that move cells farther. In this section we consider which ones will also annihilate $\Delta_{L[\boldsymbol{\alpha}]}$. In fact, due to fortuitous cancellations, we will focus on the differential operators corresponding to the complete symmetric functions.

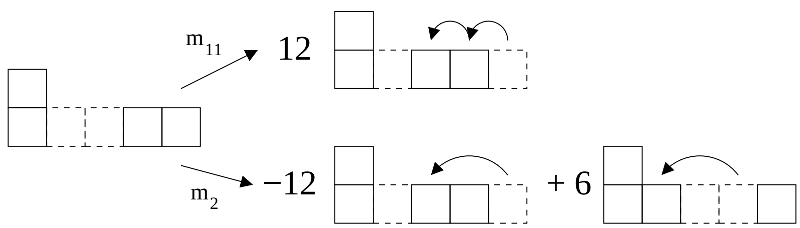

Figure 5. Illustration of the action of $h_2(\partial_{X_4}) = m_2(\partial_{X_4}) + m_{11}(\partial_{X_4})$ on $\Delta_{L[\boldsymbol{\alpha}]}$ for $\boldsymbol{\alpha} = ((0,1),(0,0),(3,0),(4,0))$.

Figure 5 illustrates the action of $h_2(\partial_X) = m_2(\partial_X) + m_{11}(\partial_X)$ on a given $\Delta_{L[\boldsymbol{\alpha}]}$. Notice that the hollow lattice diagram $\tilde{\gamma} = ((1,2),(2,0),(1,0))$ is obtained in two different ways: once through the action of $m_{11}(\partial_X)$ and once through $m_2(\partial_X)$. However, for $m_2(\partial_X)$ the cell that moves jumps over another cell. This leads to the introduction of a sign. Hence, the two $\Delta_{\tilde{\gamma}}$ that appear cancel. In fact, as Lemma 18 shows, under the action of an h_j on some $\Delta_{L[\boldsymbol{\alpha}]}$, the only term that survives is that which moves the cell $(m_1 + k_1 - 1, 0)$ (or $(0, m_2 + k_2 - 1)$, as appropriate) j spaces.

We now need to consider lattice diagrams that are subsets of hook shapes, but are not hollow. In particular, we wish to consider hollow diagrams modified by sliding certain cells closer to the origin. The amount of sliding will be described by two sequences $a_0, \ldots, a_i$ and $b_0, \ldots, b_j$ of nonincreasing, nonnegative integers. For brevity in what follows, write

$$\boldsymbol{c} = (\overline{m_2 - 1}, \ldots, \overline{1}, \underline{0}, \underline{1}, \ldots, \underline{m_1 - 1}).$$

Then, for $L_\gamma = L[\boldsymbol{\alpha}]$ with

$$\boldsymbol{\alpha} = (\overline{m_2 + k_2 + p_2 - 1}, \ldots, \overline{m_2 + k_2 - 1}, \boldsymbol{c}, \underline{m_1 + k_1 - 1}, \ldots, \underline{m_1 + k_1 + p_1 - 1}),$$

we write $\gamma[a_0, \ldots, a_i; b_0, \ldots, b_j]$ for the lattice diagram corresponding to the collection

$$(\overline{m_2 + k_2 - 1 + p_2}, \ldots, \overline{m_2 + k_2 - 1 + i - a_i}, \ldots, \overline{m_2 + k_2 - 1 + 0 - a_0}, \boldsymbol{c},$$
$$\underline{m_1 + k_1 - 1 + 0 - b_0}, \ldots, \underline{m_1 + k_1 - 1 + j - b_j}, \ldots, \underline{m_1 + k_1 - 1 + p_1}).$$

Note that due to the nonincreasing restriction on the sequences,

$$\operatorname{sgn}(\Delta_\gamma) = \operatorname{sgn}(\Delta_{\gamma[a_0,\ldots,a_i;b_0,\ldots,b_j]}).$$

An example is illustrated in Figure 6. In the figure, the left diagram is the hollow lattice diagram for $\gamma = ((2,1),(6,3),(2,2))$, while the right diagram illustrates $\gamma[2,1;4,4,3]$.

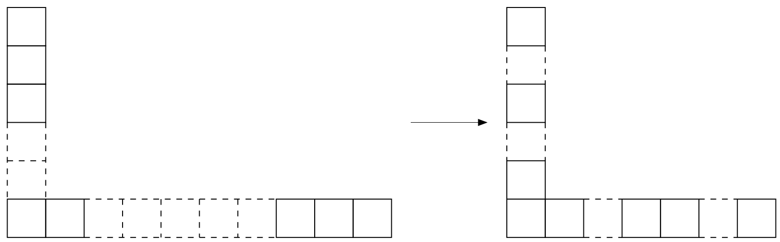

FIGURE 6. Sample square bracket notation for γ.

Lemma 18. *Let $0 \le j < k_1$ and $0 \le \ell < k_2$. Then*

$$h_j(\partial_X)\ \Delta_\gamma(X,Y) = c_j\ \Delta_{\gamma[0;j]} \quad \textit{and} \quad h_\ell(\partial_Y)\ \Delta_\gamma(X,Y) = c_\ell\ \Delta_{\gamma[\ell;0]}. \tag{49}$$

Proof. We only prove the $h_j(\partial_X)$ version as the proof of the $h_\ell(\partial_Y)$ version is effectively identical.

Recall that $h_j(X_n) = \sum_{|\lambda|=j} m_\lambda(X_n)$. Let us consider $\Delta_\gamma(X,Y)$ for $L_\gamma = L[\boldsymbol{\alpha}]$ where

$$\begin{aligned}\boldsymbol{\alpha} = \big(&\underline{0}, \underline{1}, \ldots, \underline{m_1 - 1}, \underline{m_1 + k_1 - 1}, \underline{m_1 + k_1 + p_1 - 1}, \\ &\overline{1}, \ldots, \overline{m_2 - 1}, \overline{m_2 + k_2 - 1}, \ldots, \overline{m_2 + k_2 + p_2 - 1}\big).\end{aligned} \tag{50}$$

(Note the unusual order in which we have listed the cells.) Let $\lambda = (\lambda_1, \ldots, \lambda_n) \vdash j$. View λ as an element $\boldsymbol{\lambda}$ of $\mathcal{A}^n$ under the map $\lambda_i \mapsto (\lambda_i, 0)$. It follows from Lemma 12 that

$$m_{\boldsymbol{\lambda}}(\partial_X)\ \Delta_\gamma(X,Y) = \sum_{\boldsymbol{\nu}} c_{\boldsymbol{\nu}}\ \Delta_{L[\boldsymbol{\alpha}-\boldsymbol{\nu}]}, \tag{51}$$

where the sum is over all distinct permutations $\boldsymbol{\nu}$ of $\boldsymbol{\lambda}$. As previously, we use the convention $c_{\boldsymbol{\nu}} = 0$ if $\boldsymbol{\alpha} - \boldsymbol{\nu} \notin \mathcal{A}^n$.

Consider a particular permutation $\boldsymbol{\nu}$ of $\boldsymbol{\lambda}$. If $\alpha_i - \nu_i = \alpha_j - \nu_j$ for some $i \neq j$ then $\Delta_{L[\boldsymbol{\alpha}-\boldsymbol{\nu}]} = 0$ as $\Delta_{L[\boldsymbol{\alpha}-\boldsymbol{\nu}]}$ is a determinant. Recalling the ordering of $\boldsymbol{\alpha}$ given in (50), note that if $\nu_{i,1} \neq 0$ for some $1 \le i \le m_1$, then we must have $c_{\boldsymbol{\nu}} = 0$. Therefore, without loss of generality, we may assume that $\nu_{i,1} = 0$ for $1 \le i \le m_1$ and that $\alpha_i - \nu_i \neq \alpha_j - \nu_j$ for $i \neq j$.

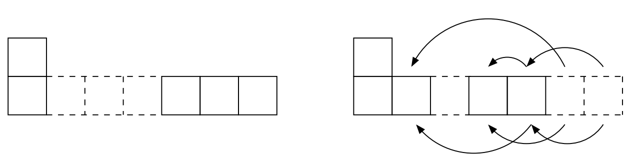

FIGURE 7. Illustration of the involution described in the proof of Lemma 18.

Let $\boldsymbol{\mu}$ be given by $\mu_{m_1+1} = (j,0)$ and $\mu_i = (0,0)$ for $i \neq m_1+1$. We now proceed to define an involution on the $\boldsymbol{\nu}$ whose only fixed point is $\boldsymbol{\mu}$. So assume $\boldsymbol{\nu} \neq \boldsymbol{\mu}$.

Define $p < r$ to index the smallest two terms (with respect to $<_{\mathcal{A}}$) of

$$\alpha_{m_1+1} - \nu_{m_1+1}, \alpha_{m_1+2} - \nu_{m_1+2}, \ldots, \alpha_{m_1+p_1+1} - \nu_{m_1+p_1+1}. \tag{52}$$

The indices p and r index the two boxes of $\Delta_{L[\boldsymbol{\alpha}]}$ not contiguous with the cell $(0,0)$ that have moved farthest to the left upon subtraction of $\boldsymbol{\nu}$. Set

$$q = (\alpha_{r,1} - \nu_{r,1}) - (\alpha_{p,1} - \nu_{p,1}) . \tag{53}$$

We now define

$$\boldsymbol{\nu}' = g(\boldsymbol{\nu}) = \Big[\nu_1, \nu_2, \ldots, \underline{\nu_{p,1} - q}, \ldots, \underline{\nu_{r,1} + q}, \ldots, \nu_n\Big] . \tag{54}$$

The left-hand picture in Figure 7 illustrates $\Delta_{L[\boldsymbol{\alpha}]}$ for $\boldsymbol{\alpha} = (\underline{0}, \underline{4}, \underline{5}, \underline{6}, \overline{1})$. The right-hand picture illustrates $\Delta_{L[\boldsymbol{\alpha}-\boldsymbol{\nu}]}$ for $\boldsymbol{\nu} = (\underline{0}, \underline{3}, \underline{2}, \underline{2}, \underline{0})$ via the bottom triple of arrows ($p = 2$, $r = 3$, $\underline{1} = \alpha_2 - \nu_2 < \alpha_3 - \nu_3 = \underline{3}$, and $q = 2$) and $\Delta_{L[\boldsymbol{\alpha}-\boldsymbol{\nu}']}$ for $\boldsymbol{\nu}' = (\underline{0}, \underline{1}, \underline{4}, \underline{2}, \underline{0})$ via the top triple of arrows. When viewed as sets, $\boldsymbol{\alpha} - \boldsymbol{\nu}$ equals $\boldsymbol{\alpha} - \boldsymbol{\nu}'$; they differ only in order.

We have that

$$\Delta_{L[\boldsymbol{\alpha}-\boldsymbol{\nu}]} = -\Delta_{L[\boldsymbol{\alpha}-\boldsymbol{\nu}']}, \tag{55}$$

since $\boldsymbol{\alpha} - \boldsymbol{\nu}$ and $\boldsymbol{\alpha} - \boldsymbol{\nu}'$ differ by a transposition. Now $g(g(\boldsymbol{\nu})) = \boldsymbol{\nu}$. As desired, this function g yields a sign-reversing involution between all the terms $\Delta_{L[\boldsymbol{\alpha}-\boldsymbol{\nu}]}$ in equation (51) except for the unique $\Delta_{L[\boldsymbol{\alpha}-\boldsymbol{\mu}]}$. It is not difficult to see that the corresponding coefficients $c_{\boldsymbol{\nu}'}$ in the expansion of $m_\lambda(\partial_X)\Delta_\gamma(X,Y)$ in terms of the $\Delta_{L[\boldsymbol{\alpha}-\boldsymbol{\nu}]}$ satisfy $c_{\boldsymbol{\nu}} = c_{\boldsymbol{\nu}'}$. Thus, all the terms in equation (51) cancel out except $\Delta_{L[\boldsymbol{\alpha}-\boldsymbol{\mu}]}$. This gives the lemma. □

Recall that $\Delta_\gamma(X,Y) = 0$ if two of the entries in L_γ are identical. Thus, if $j \geq k_1$ or $\ell \geq k_2$, then $\Delta_{\gamma[0;j]} = 0$ or $\Delta_{\gamma[\ell;0]} = 0$, respectively. It follows that

Corollary 19. *$h_{k_1+i}(X_n) \in \mathcal{I}_\gamma(X_n, Y_n)$ for $i \geq 0$ and $h_{k_2+h}(Y_n) \in \mathcal{I}_\gamma(X_n, Y_n)$ for $h \geq 0$.*

Corollary 19 provides information about certain elements that must be in the ideal $\mathcal{I}_\gamma(X_n, Y_n)$. As such, we will use it to define a sub-ideal $\mathcal{K}_\gamma(X_n, Y_n)$ of $\mathcal{I}_\gamma(X_n, Y_n)$. Specifically, set $\mathcal{K}_\gamma(X_n, Y_n)$ to be the ideal in $\mathbb{C}[X_n, Y_n]$ generated by the generators of $\mathcal{H}_\gamma(X_n, Y_n)$ along with

$$(h_{k_1}(X_n), \ldots, h_{k_1+p_1+1}(X_n), \ldots, h_{k_2}(Y_n), \ldots, h_{k_2+p_2+1}(Y_n), \ldots) . \tag{56}$$

As it turns out, $\mathcal{K}_\gamma(X_n, Y_n)$ is a finitely generated ideal in $\mathbb{C}[X_n, Y_n]_{\mathcal{H}_\gamma}$. To this end, define $\mathcal{J}_\gamma(X_n, Y_n)$ to be the sub-ideal of $\mathcal{K}_\gamma(X_n, Y_n)$ generated by the generators of $\mathcal{H}_\gamma(X_n, Y_n)$ along with

$$\{h_{k_1}(X_n), \ldots, h_{k_1+p_1}(X_n), h_{k_2}(Y_n), \ldots, h_{k_2+p_2}(Y_n)\} . \tag{57}$$

It follows from Corollary 19 that $\mathcal{J}_\gamma \subset \mathcal{K}_\gamma \subset \mathcal{I}_\gamma$.

Lemma 20. *$\mathcal{K}_\gamma(X_n, Y_n) \equiv \mathcal{J}_\gamma(X_n, Y_n) \pmod{\mathcal{H}_\gamma(X_n, Y_n)}$.*

Proof. A standard result (cf. [14, pg. 21]) is that

$$\sum_{r=0}^{n} (-1)^r \; e_r(X_n) \; h_{n-r}(X_n) = 0. \tag{58}$$

We prove that $h_{k_1+p_1+a}(X_n) \equiv 0 \pmod{\mathcal{J}_\gamma(X_n, Y_n)}$ for $a \geq 1$ by induction on a. For $a \geq 1$, we have

$$h_{k_1+p_1+a}(X_n) = \sum_{r=1}^{p_1+1} (-1)^{r+1}\ e_r(X_n)\ h_{k_1+p_1+a-r}(X_n) + \sum_{r=p_1+2}^{k_1+p_1+a} (-1)^{r+1}\ e_r(X_n)\ h_{k_1+p_1+a-r}(X_n). \tag{59}$$

Recall from (34) and (46) that $e_{p_1+j}(X_n) \in \mathcal{H}_\gamma(X_n, Y_n)$ for $j \geq 2$. So the second sum of (59) is in $\mathcal{H}_\gamma(X_n, Y_n) \subset \mathcal{J}_\gamma(X_n, Y_n)$. On the other hand, by the definition of $\mathcal{J}_\gamma(X_n, Y_n)$, $h_{k_1+i} \in \mathcal{J}_\gamma(X_n, Y_n)$ for $0 \leq i \leq p_1$. So the first sum of (59) is in $\mathcal{J}_\gamma(X_n, Y_n)$ as well. This proves the claim for $a = 1$. The claim for $a > 1$ thereby follows by the obvious induction hypothesis. Similar arguments can be made about $h_{k_2+p_2+a}(Y_n)$, for $a \geq 1$. □

Our next goal is to show that the collection that generates $\mathcal{J}_\gamma(X_n, Y_n)$ in equation (57) is itself algebraically independent in $\mathbb{C}[X_n, Y_n]_{\mathcal{H}_\gamma}$.

Theorem 21. *The collection*

$$\{h_{k_1}(X_n), h_{k_1+1}(X_n), \ldots, h_{k_1+p_1}(X_n), h_{k_2}(Y_n), h_{k_2+1}(Y_n), \ldots, h_{k_2+p_2}(Y_n)\} \tag{60}$$

is algebraically independent in the ring $\mathbb{C}[X_n, Y_n]_{\mathcal{H}_\gamma}$.

The reader is advised to follow Example 22 while reading the proof.

Proof. The theorem is equivalent to the statement that there is no nontrivial polynomial P over $\mathbb{C}$ in $p_1 + p_2 + 2$ variables such that

$$P(h_{k_1}(X_n), \ldots, h_{k_1+p_1}(X_n), h_{k_2}(Y_n), \ldots, h_{k_2+p_2}(Y_n)) \in \mathcal{H}_\gamma(X_n, Y_n). \tag{61}$$

Since the x_i and y_i are true indeterminates, such a relation would have to continue to hold upon the specialization $y_1 = \cdots = y_n = 0$. Hence, it suffices to show that there is no nontrivial polynomial Q such that

$$Q(h_{k_1}(X_n), \ldots, h_{k_1+p_1}(X_n)) \in \mathcal{H}_\gamma(X_n, Y_n). \tag{62}$$

So, to prove the theorem, we assume that such a Q does exist and obtain a contradiction. Specifically, we will show that if the h_i in question are algebraically dependent in $\mathcal{H}_\gamma(X_n, Y_n)$, then the e_i of (45) are algebraically dependent in $\mathcal{G}_\gamma(X_n, Y_n)$, in direct contradiction with Lemma 15.

A relation such as (62) can be rewritten as

$$\sum_\lambda c_\lambda\ h_\lambda \in \mathcal{H}_\gamma(X_n, Y_n), \tag{63}$$

where each λ is a partition with parts of length chosen from the collection $\{k_1, k_1 + 1, \ldots, k_1 + p_1\}$. We assume that $c_\lambda = 0$ for any λ with $h_\lambda \in \mathcal{H}_\gamma(X_n, Y_n)$ and that there exists some $c_\lambda \neq 0$.

Since the monomial symmetric functions are a basis of the ring of symmetric functions, any such h_λ can be expanded in terms of the e_μ. In fact, this can be done explicitly as follows (cf. [14, pg. 107]).

Define a *domino* to be a set of horizontally consecutive squares in a Ferrers diagram. For $\lambda, \mu \vdash n$, a *domino tabloid of shape* λ *and type* μ is a tiling of $\mathrm{dg}(\lambda)$ with dominoes of length $\mu_1, \mu_2, \ldots$. We consider dominoes of the same length to be

indistinguishable. Let $d_{\lambda,\mu}$ denotes the number of domino tabloids of shape λ and type μ. Then we can write

$$h_\lambda(X_n) = \sum_{\mu} (-1)^{|\mu|-\ell(\mu)} d_{\lambda,\mu}\, e_\mu(X_n). \tag{64}$$

Recall that e_μ is in $\mathcal{G}_\gamma(X_n, Y_n)$ (respectively, $\mathcal{H}_\gamma(X_n, Y_n)$) whenever μ has a part greater than or equal to $m_1 + p_1 + 1$ (respectively, $p_1 + 2$). Combining (64) and (63), we find that

$$\sum_{\substack{\mu \\ \mu_1 \le p_1+1}} (-1)^{|\mu|-\ell(\mu)} \left(\sum_{\lambda} c_\lambda d_{\lambda,\mu} \right) e_\mu(X_n) \equiv 0 \pmod{\mathcal{H}_\gamma(X_n, Y_n)}, \tag{65}$$

or, equivalently, that

$$\begin{aligned} \sum_{\substack{\mu \\ \mu_1 \le p_1+1}} (-1)^{|\mu|-\ell(\mu)} \left(\sum_{\lambda} c_\lambda d_{\lambda,\mu} \right) e_\mu(X_n) & \\ - \sum_{\substack{\mu \\ p_1+2 \le \mu_1 \le p_1+m_1}} a_\mu e_\mu(X_n) &\equiv 0 \pmod{\mathcal{G}_\gamma(X_n, Y_n)}. \end{aligned} \tag{66}$$

for some constants a_μ. If one of the $a_\mu \neq 0$ or one of the $\sum_\lambda c_\lambda d_{\lambda,\mu} \neq 0$ for some μ, then we have a contradiction (asserted by Corollary 15) of the algebraic independence in $\mathbb{C}[X_n, Y_n]_{\mathcal{G}_\gamma}$ of the set $\{e_i\}_{1 \le i \le p_1+m_1}$. We will, in fact, show that at least one of the $\sum_\lambda c_\lambda d_{\lambda,\mu}$ is nonzero.

For a given λ, there is a maximum (with respect to lexicographic order) μ with $d_{\lambda,\mu} \neq 0$ and $e_\mu \notin \mathcal{H}_\gamma(X_n, Y_n)$. It is given by

$$L(\lambda) = \left((p_1+1)^{\alpha_{p_1+1}}, (p_1)^{\alpha_{p_1}}, (p_1-1)^{\alpha_{p_1-1}}, \ldots, (1)^{\alpha_1}\right). \tag{67}$$

Here, for $1 \le f \le p_1$, α_f equals the number of entries in

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)}) \tag{68}$$

congruent to $f \pmod{p_1+1}$ and

$$\alpha_{p_1+1} = \sum_{r=1}^{\ell(\lambda)} \left\lfloor \frac{\lambda_r}{p_1+1} \right\rfloor \tag{69}$$

($\lfloor x \rfloor$ denotes the *floor of* x). In particular, modulo $\mathcal{H}_\gamma(X_n, Y_n)$, we can rewrite (64) as

$$h_\lambda(X_n) \equiv \pm d_{\lambda,L(\lambda)} e_{L(\lambda)}(X_n) + \sum_{\mu <_{\mathrm{rlex}} L(\lambda)} (-1)^{|\mu|-\ell(\mu)} d_{\lambda,\mu}\, e_\mu(X_n) \tag{70}$$

with $d_{\lambda,L(\lambda)} \neq 0$.

Furthermore, given μ, there is a unique partition λ (possibly with $c_\lambda = 0$) for which μ is the maximum element appearing in (64) with $d_{\lambda,\mu} \neq 0$ for some $\tilde{\lambda}$. To see this, note that each α_f (when $0 \le f \le p_1$) gives the number of $\tilde{\lambda}_j$ congruent to $f \pmod{p_1+1}$. Since $k_1 \le \tilde{\lambda}_j \le k_1 + p_1$ this uniquely identifies $\tilde{\lambda}_j$. Each such $\tilde{\lambda}_j$ requires $\left\lfloor \frac{\tilde{\lambda}_j}{p_1+1} \right\rfloor$ dominoes of length p_1+1 to finish out the row. Any remaining p_1+1 will be used to construct the remaining rows of $\tilde{\lambda}$.

To finish the proof, pick from (63) the partition $\tilde{\lambda}$ occurring with $c_{\tilde{\lambda}} \neq 0$ for which $L(\tilde{\lambda})$ is maximal. Certainly $d_{\tilde{\lambda},L(\tilde{\lambda})} \neq 0$. However, by this choice of $\tilde{\lambda}$, $d_{\lambda,L(\tilde{\lambda})} = 0$ or $c_\lambda = 0$ for $\lambda \neq \tilde{\lambda}$. Hence, the coefficient of $e_{L(\tilde{\lambda})}$ in (66) is nonzero as desired. □

Example 22. Fix $\gamma = ((4,1),(7,0),(2,0))$. In this example we consider the expansion of $h_\lambda = h_{(9,9,9,8,7,7)}$ in terms of the monomial symmetric functions modulo $\mathcal{H}_\gamma(X_n, Y_n)$. In particular, we construct $L(\lambda)$ and show that λ is recoverable from it.

From $p_1 = 2$ it follows that $L((9^3, 8, 7^2)) = (3^{15}, 2, 1^2)$. The dominoes on each of the rows of length 7 and 8 can be placed in three different ways, so we find $d_{(9^3,8,7^2),(3^{15},2,1^2)} = 27$. Then, following (70), $h_{(9^3,8,7^2)}(X_n)$ is congruent to

$$-27e_{(3^{15},2,1^2)}(X_n) + \sum_{\mu <_{\text{lex}} (3^{15},2,1^2)} (-1)^{|\mu|-\ell(\mu)} d_{(9^3,8,7^2),\mu} e_\mu(X_n) \tag{71}$$

modulo $\mathcal{H}_\gamma(X_n, Y_n)$. On the other hand, given $L(\lambda) = (3^{15}, 2, 1^2)$, we can recover λ as follows. We know that the parts of λ must all be between k_1 and $k_1 + p_1$; 7 and 9 in this case. The two 1's in $L(\lambda)$ tell us that there must be two parts of λ of length 1 modulo $p_1 + 1 = 3$ (i.e., two rows of length 7). Similarly, we compute that there is a unique row of length 8. These three rows account for six of the fifteen length 3 parts of $L(\lambda)$. The remaining nine length 3 parts must together comprise the parts of length 9. Hence there are three of them and we recover $\lambda = (9^3, 8, 7^2)$ as desired. This completes our example.

Theorem 21 implies the following.

Corollary 23. *The Hilbert Series of $\mathbb{C}[X_n, Y_n]_{\mathcal{J}_\gamma}$ is given by*

$$\begin{aligned} \mathcal{H}(\mathbb{C}[X_n, Y_n]_{\mathcal{J}_\gamma}) &= \frac{(t)_{k_1+p_1}}{(t)_{k_1-1}(t)_{p_1+1}} \frac{(q)_{k_2+p_2}}{(t)_{k_2-1}(t)_{p_2+1}} \sum_{b \in \mathcal{B}_\gamma} t^{|X(b)|} q^{|Y(b)|} \\ &= \begin{bmatrix} p_1 + k_1 \\ p_1 + 1 \end{bmatrix}_t \begin{bmatrix} p_2 + k_2 \\ p_2 + 1 \end{bmatrix}_q \sum_{b \in \mathcal{B}_\gamma} t^{|X(b)|}\ q^{|Y(b)|}. \end{aligned} \tag{72}$$

8. The ideals $\mathcal{J}_\gamma(X_n, Y_n) = \mathcal{I}_\gamma(X_n, Y_n)$.

We need to identify the generators of $\mathcal{I}_\gamma(X_n, Y_n)$ (recall equation (4)). Specifically, we want to prove that it is finitely generated by a collection of complete symmetric functions in the ring $\mathbb{C}[X_n, Y_n]_{\mathcal{H}_\gamma}$. The goal is to show that the ideals $\mathcal{J}_\gamma$ and $\mathcal{I}_\gamma$ are equal in $\mathbb{C}[X_n, Y_n]_{\mathcal{H}_\gamma}$. Since $\mathcal{J}_\gamma \subseteq \mathcal{I}_\gamma$, we will do this by constructing a linearly independent set in $\mathbb{C}[X_n, Y_n]_{\mathcal{I}_\gamma}$ that gives the Hilbert series in equation (72). Set

$$Q_{k,p} = \{(q_1, \ldots, q_p) \in \mathbb{N}^p : \text{ the } q_i \text{ are nonincreasing and } q_1 \leq k\}. \tag{73}$$

Observe that $|Q_{k,p}| = \binom{p+k}{p}$. We can now define the collection of polynomials $\mathcal{BB}_\gamma$ that will turn out to be the required basis for $\mathbb{C}[X_n, Y_n]_{\mathcal{I}_\gamma}$. (Recall that $\mathcal{B}_\gamma$ is defined in equation (39).)

$$\mathcal{BB}_\gamma = \{h_q(X_n) h_{q'}(Y_n)\ b : b \in \mathcal{B}_\gamma, q \in Q_{k_1-1,p_1+1}, q' \in Q_{k_2-1,p_2+1}\}. \tag{74}$$

It follows from the remark above that the number of elements in $\mathcal{BB}_\gamma$ is

$$n!\binom{p_1+k_1}{p_1+1}\binom{p_2+k_2}{p_2+1}. \tag{75}$$

Furthermore, note that

$$\sum_{b\in\mathcal{BB}_\gamma} t^{|X(b)|}\ q^{|Y(b)|} = \begin{bmatrix} p_1+k_1 \\ p_1+1 \end{bmatrix}_t \begin{bmatrix} p_2+k_2 \\ p_2+1 \end{bmatrix}_q \sum_{b\in\mathcal{B}_\gamma} t^{|X(b)|}\ q^{|Y(b)|}, \tag{76}$$

which equals the summation found in equation (72).

In Theorem 25 we will consider the action of differential operators in the complete symmetric functions on the determinants Δ_γ. This theorem generalizes Lemma 18. The below example gives a sample computation in this spirit.

Example 24. By Lemma 18, we have

$$\begin{aligned} h_{3,2}(\partial_X)\ \Delta_{L[(\underline{0},\underline{1},\underline{2},\underline{3},\underline{9},\underline{10},\underline{11})]} &= h_2(\partial_X)\ h_3(\partial_X)\ \Delta_{L[(\underline{0},\underline{1},\underline{2},\underline{3},\underline{9},\underline{10},\underline{11})]} \\ &= ((m_2+m_{1,1})(\partial_X))\,(9\cdot 8\cdot 7)\Delta_{L[(\underline{0},\underline{1},\underline{2},\underline{3},\underline{6},\underline{10},\underline{11})]} \end{aligned} \tag{77}$$

equal to

$$\begin{aligned} &\frac{11!}{6!}\ \Delta_{L[(\underline{0},\underline{1},\underline{2},\underline{3},\underline{6},\underline{10},\underline{9})]} + \frac{11!}{6!}\ \Delta_{L[(\underline{0},\underline{1},\underline{2},\underline{3},\underline{6},\underline{9},\underline{10})]} + 9\frac{10!}{6!}\ \Delta_{L[(\underline{0},\underline{1},\underline{2},\underline{3},\underline{6},\underline{8},\underline{11})]} \\ &\qquad + \frac{10!}{5!}\ \Delta_{L[(\underline{0},\underline{1},\underline{2},\underline{3},\underline{5},\underline{9},\underline{11})]} + \frac{9!}{4!}\ \Delta_{L[(\underline{0},\underline{1},\underline{2},\underline{3},\underline{4},\underline{10},\underline{11})]}. \end{aligned} \tag{78}$$

(We have omitted those $\Delta_{L[\boldsymbol{\alpha}]}$ that have repeated entries and are thus identically equal to zero. We have also included exact coefficients even those these are not given by Lemma 18.) Note that the first and second terms cancel as they differ by the transposition of adjacent elements. This completes our example.

Suppose the sequences $\boldsymbol{\beta}$ and $\boldsymbol{\delta}$ correspond to $\gamma' = \gamma[a_0,\ldots,a_i;b_0,\ldots b_j]$ and $\gamma'' = \gamma[a_0',\ldots,a_i';b_0',\ldots b_j']$, respectively, for some triple γ and nonincreasing sequences (a_ℓ), (b_ℓ), (a_ℓ') and (b_ℓ'). We write $\boldsymbol{\beta} >_{\text{cont}} \boldsymbol{\delta}$ (or $\gamma' >_{\text{cont}} \gamma''$) whenever

(1) $(a_0,\ldots,a_i) >_{\text{lex}} (a_0',\ldots,a_i')$; or
(2) if $(a_0,\ldots,a_i) = (a_0',\ldots,a_i')$ then $(b_0,\ldots,b_j) >_{\text{lex}} (b_0',\ldots,b_j')$.

Theorem 25. *If*

$$\begin{aligned} q &= (q_1,q_2,\ldots,q_{p_1+1}) \in Q_{k_1-1,p_1+1} \quad \textit{and} \\ q' &= (q_1',q_2',\ldots,q_{p_2+1}') \in Q_{k_2-1,p_2+1}, \end{aligned} \tag{79}$$

then we have

$$h_q(\partial_X)\ h_{q'}(\partial_Y)\ \Delta_\gamma(X,Y) = c_{\tilde\gamma}\ \Delta_{\tilde\gamma} + \sum_{\gamma''>_{\text{cont}}\tilde\gamma} c_{\gamma''}\ \Delta_{\gamma''} \tag{80}$$

where $c_{\tilde\gamma} > 0$ *and*

$$\tilde\gamma = \gamma[q_1,q_2,\ldots,q_{p_1+1};q_1',q_2',\ldots,q_{p_2+1}']. \tag{81}$$

Proof. This follows from the argument in the proof of Lemma 18. Recall that since a complete symmetric function h_j can be written as a nonnegative linear combination of monomial symmetric functions, the action of $h_j(\partial_X)$ can be viewed as the movement of boxes to the left such that the distances traveled sum to j. In the proof of that lemma, we defined an involution that (among other things) canceled the contributions coming from any m_μ with $\mu \neq (j)$.

In the situation of this theorem, we can proceed analogously. Here, though, we need to allow for the possibility that a box moved by an h_{q_i} also gets moved by an h_{q_j} for some $j \neq i$. However, any such resulting γ'' will satisfy $\gamma'' >_{\text{cont}} \tilde{\gamma}$. □

The primary goal of this paper is to prove the following theorem.

Theorem 26. *The collection $\mathcal{BB}_\gamma$ is linearly independent in $\mathbb{C}[X_n, Y_n]_{\mathcal{I}_\gamma}$. Hence, by the equality of* (72) *and* (76)*, $\mathcal{BB}_\gamma$ forms a basis for $\mathbb{C}[X_n, Y_n]_{\mathcal{I}_\gamma}$.*

By [1, Theorem 2.1], the subspace of $\mathcal{B}_\gamma$ spanned by bipermanents $[T, V]_{\text{per}}$ of a given shape λ and given V carries a copy of the irreducible S^λ. It is this fact that lets us conclude Theorem 1 from the Hilbert series (72) of Corollary 23.

Note that it is enough to show that the collection

$$\{b(\partial_X, \partial_Y)\ \Delta_\gamma(X, Y) : b \in \mathcal{BB}_\gamma\} \tag{82}$$

is linearly independent in $\mathbb{C}[X_n, Y_n]$ since

$$\begin{aligned} \sum_{j=1}^{i} c_j b_j \equiv 0 \pmod{\mathcal{I}_\gamma(X_n, Y_n)} &\Longleftrightarrow \left(\sum_{j=1}^{i} c_j b_j\right)(\partial_X, \partial_Y)\ \Delta_\gamma(X, Y) = 0 \\ &\Longleftrightarrow \sum_{j=1}^{i} (c_j\ b_j(\partial_X, \partial_Y)\ \Delta_\gamma(X, Y)) = 0. \end{aligned} \tag{83}$$

To do this, however, we need the following lemma. A proof can be found in [3] (see equations (6.5) and (6.6) in Theorem 6.2), however, there is a sign missing, so we include a proof here. The lemma lets us expand the action of a bipermanent on a determinant associated to a lattice diagram. To state the lemma succinctly, we first associate to the bipermanent in question a family of $\mathcal{A}$-filled diagrams. So, consider a bitableau $(T, C) \in \Theta_n$ and set $S = \text{std}(C)$. Write $\iota = \iota_{T,S}$ for the map that takes i to s_i^T. For any permutation $\phi \in S_n$, we then define an $\mathcal{A}'$-filled diagram $E_\phi^{\boldsymbol{\alpha}}$ by placing $\alpha_{\phi(\iota(i))} - c_i^T$ in the cell containing i in T.

Lemma 27. *Let $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_n]$ and $(T, C) \in \Theta_n$. Then*

$$[T, C]_{\text{per}}(\partial_X, \partial_Y)\ \Delta_{L[\boldsymbol{\alpha}]} = \text{sgn}(\iota) \sum_{\phi \in S_n} \text{sgn}(\phi)\ d_\phi\ [T^t, (E_\phi^{\boldsymbol{\alpha}})^t]_{\det}, \tag{84}$$

for the $\mathcal{A}'$-filled diagrams $E_\phi^{\boldsymbol{\alpha}}$ defined above and integers $d_\phi \geq 0$. We make the convention that $d_\phi = 0$ if any entry of $E_\phi^{\boldsymbol{\alpha}} \in \mathcal{A}' - \mathcal{A}$ or if $[T^t, (E_\phi^{\boldsymbol{\alpha}})^t]_{\det} = 0$; otherwise $d_\phi > 0$.

Note that $[T^t, (E_\phi^{\boldsymbol{\alpha}})^t]_{\det} = 0$ when there is a repetition in some row of $E_\phi^{\boldsymbol{\alpha}}$.

Proof. Define $q_{T,C} = q_{T,C}(\partial_X, \partial_Y)$ to be the monomial $\partial_{z_1}^{c_1^T} \partial_{z_2}^{c_2^T} \cdots \partial_{z_n}^{c_n^T}$. Then $[T, C]_{\text{per}}$ can be expanded as $\sum_{\sigma \in R_T} \sigma(q_{T,C})$. Expanding $\Delta_{L[\boldsymbol{\alpha}]}$ as well, we have

the following expression for the left-hand side of (84):

$$
\begin{aligned}
[T,C]_{\mathrm{per}}(\partial_X,\partial_Y)\Delta_{L[\boldsymbol{\alpha}]} &= \sum_{\sigma\in R_T}\sum_{\tau\in S_n}\operatorname{sgn}(\tau)(\sigma q_{T,C})\prod_{j=1}^{n} z_j^{\alpha_{\tau^{-1}(j)}} \\
&= \sum_{\sigma\in R_T}\sum_{\tau\in S_n}\operatorname{sgn}(\tau)\,\sigma\left(q_{T,C}\prod_{j=1}^{n} z_j^{\alpha_{\sigma(\tau^{-1}(j))}}\right) \\
&= \sum_{\phi\in S_n}\sum_{\sigma\in R_T}\operatorname{sgn}(\iota\circ\phi\circ\sigma)\,\sigma\left(q_{T,C}\prod_{j=1}^{n} z_j^{\alpha_{\phi(\iota(j))}}\right).
\end{aligned} \tag{85}
$$

As an illustration of the equality between the first two lines, consider

$$((1,5,3)\partial_{x_5}^4)x_3^{\alpha_3} = \alpha_3^{\underline{4}}x_3^{\alpha_3-4} = (1,5,3)(\partial_{x_5}^4 x_5^{\alpha_3}). \tag{86}$$

In going from the second to the third lines, we set $\phi = \sigma\circ\tau^{-1}\circ\iota^{-1}$, used the fact that the sign of any permutation is the sign of its inverse, and noted that as τ runs over S_n, so does ϕ.

For each ϕ, we would like to interpret the sum over σ as a bideterminant $[T^t,(E_\phi^{\boldsymbol{\alpha}})^t]_{\mathrm{det}}$. Formally this makes sense as bideterminants are signed sums over the elements in the column stabilizer of some filled diagram. Here we have a signed sum over a row stabilizer; hence we consider transposes. If $q_{T,C}(\partial_X,\partial_Y)$ were not in the equation, we would use $\alpha_{\phi(\iota(i))}$ as our entry in $E_\phi^{\boldsymbol{\alpha}}$ corresponding to i in T. (The ι accounts for the fact that we are not assuming $S=T$.) Up to the multiplicative constants d_ϕ, the action of $q_{T,C}(\partial_X,\partial_Y)$ is to subtract c_i^T. This is consistent with the statement of the lemma. □

Example 28. We illustrate Lemma 27 with the following simple computation. In this example, the identity is the only ϕ for which the entries of $E_\phi^{\boldsymbol{\alpha}}$ are all in $\mathcal{A}$. Also, ι is given in cycle notation by $(2,4,3)$ which yields $\operatorname{sgn}(\iota)=1$. In the second-to-last line, the subtraction of filled diagrams should be interpreted entrywise.

$$
\begin{aligned}
&\left[\begin{array}{|c|c|}\cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 1 & 2 \\ \hline\end{array}\;,\;\begin{array}{|c|c|}\cline{1-1} \underline{1} & \multicolumn{1}{c}{} \\ \cline{1-1} \underline{\underline{0}} & \multicolumn{1}{c}{} \\ \hline \overline{1} & \underline{2} \\ \hline\end{array}\right]_{\mathrm{per}}(\partial_X,\partial_Y)\Delta_{L[(\overline{2},\underline{0},\underline{1},\underline{3})]} \\
&\quad= \left(\partial_{y_1}\partial_{x_2}^2\partial_{x_4} + \partial_{y_2}\partial_{x_1}^2\partial_{x_4}\right)\det\begin{pmatrix} y_1^2 & y_2^2 & y_3^2 & y_4^2 \\ 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 \end{pmatrix} \\
&\quad= 12y_1x_2 - 12y_2x_1 = 12\left[\begin{array}{|c|c|c|}\cline{1-1} 2 & \multicolumn{2}{c}{} \\ \hline 1 & 3 & 4 \\ \hline\end{array}\,,\,\begin{array}{|c|c|c|}\cline{1-1} \underline{\underline{3}} & \multicolumn{2}{c}{} \\ \hline \overline{2} & \underline{0} & \underline{1} \\ \hline\end{array} - \begin{array}{|c|c|c|}\cline{1-1} \underline{\underline{2}} & \multicolumn{2}{c}{} \\ \hline \overline{1} & \underline{0} & \underline{1} \\ \hline\end{array}\right]_{\mathrm{det}} \\
&\quad= 12\left[\begin{array}{|c|c|c|}\cline{1-1} 2 & \multicolumn{2}{c}{} \\ \hline 1 & 3 & 4 \\ \hline\end{array}\,,\,\begin{array}{|c|c|c|}\cline{1-1} \underline{\underline{1}} & \multicolumn{2}{c}{} \\ \hline \overline{1} & \underline{0} & \underline{0} \\ \hline\end{array}\right]_{\mathrm{det}}.
\end{aligned} \tag{87}
$$

For U an $\mathcal{A}$-filled diagram, we define

$$
\begin{aligned}
\mathrm{con}_{\mathrm{X}}(U) &= [u_{m_2+p_2+2}, u_{m_2+p_2+3},\dots,u_n], \\
\mathrm{con}_{\mathrm{Y}}(U) &= [u_1,u_2,\dots u_{m_2+p_2}],
\end{aligned} \tag{88}
$$

where the square brackets indicate that we have arranged the elements of the sets in increasing order. Let $(T_1, U_1), (T_2, U_2)$ be bitableaux. We define

$$(T_1, U_1) <_{\text{bitab}} (T_2, U_2) \tag{89}$$

according to the following tiebreakers (recall that $\mathrm{cs}(U)$ denotes the column sequence of U; cf. Example 5):

(1) $\mathrm{sh}(T_1^t) <_{\text{lex}} \mathrm{sh}(T_2^t)$;
(2) $\mathrm{con}_{\mathrm{X}}(U_1) >_{\text{lex}} \mathrm{con}_{\mathrm{X}}(U_2)$;
(3) $\mathrm{con}_{\mathrm{Y}}(U_1) <_{\text{lex}} \mathrm{con}_{\mathrm{Y}}(U_2)$;
(4) $\mathrm{cs}(U_1) <_{\text{lex}(\mathcal{A})} \mathrm{cs}(U_2)$;
(5) $\mathrm{cs}(T_1) >_{\text{lex}(\mathcal{A})} \mathrm{cs}(T_2)$.

We are now ready to prove that $\mathcal{BB}_\gamma$ is a basis for $\mathbb{C}[X_n, Y_n]_{\mathcal{I}_\gamma}$. As outlined in (83), it suffices to prove the following theorem.

Theorem 29. *Let $T, U \in SYT_n$, $q = (q_1, q_2, \ldots, q_{p_1+1}) \in Q_{k_1-1, p_1+1}$ and $q' = (q'_1, q'_2, \ldots, q'_{p_2+1}) \in Q_{k_2-1, p_2+1}$. Set $C = C_\gamma(U)$. Then*

$$h_q(\partial_X)\ h_{q'}(\partial_Y)\ [T, C]_{\text{per}}(\partial_X, \partial_Y)\ \Delta_\gamma = d\ [T^t, P^t]_{\det} + \sum_{(S,W)} d_{S,W} [S^t, W^t]_{\det}, \tag{90}$$

where $d \neq 0$, $P = E_\varepsilon^{\boldsymbol{\beta}}$, and the sum is over all $(S, W) \in \Theta_n$ with $(S, W) >_{\text{bitab}} (T, P)$.

Before presenting the proof, we illustrate (90) with an explicit computation.

Example 30. Let γ derive from the collection of boxes $\boldsymbol{\alpha} = (\overline{4}, \overline{3}, \underline{0}, \underline{3}, \underline{4}, \underline{5})$. Set $T = \begin{array}{|c|c|}\hline 4 & 6 \\ \hline 3 & 5 \\ \hline 1 & 2 \\ \hline \end{array}$ and $U = \begin{array}{|c|c|}\hline 5 & 6 \\ \hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array}$. Then $C = C_\gamma(U) = \begin{array}{|c|c|}\hline \underline{2} & \underline{2} \\ \hline \underline{0} & \underline{1} \\ \hline \overline{1} & \underline{0} \\ \hline \end{array}$ and $\iota = (2,3)(4,5)$. We consider the expansion of

$$\begin{aligned} h_{1,1}(\partial_X) & h_{2,1}(\partial_Y)[T, C]_{\text{per}}(\partial_X, \partial_Y)\Delta_\gamma \\ &= [T, C]_{\text{per}}(\partial_X, \partial_Y) h_{1,1}(\partial_X) h_{2,1}(\partial_Y)\Delta_\gamma \\ &= [T, C]_{\text{per}}(\partial_X, \partial_Y) \left(c_{\boldsymbol{\beta}}\ \Delta_{L[\boldsymbol{\beta}]} + \sum_{\boldsymbol{\delta} >_{\text{cont}} \boldsymbol{\beta}} c_{\boldsymbol{\delta}}\ \Delta_{L[\boldsymbol{\delta}]} \right) \\ &= c_{\boldsymbol{\beta}}\ [T, C]_{\text{per}}(\partial_X, \partial_Y)\ \Delta_{L[\boldsymbol{\beta}]} + \sum_{\boldsymbol{\delta} >_{\text{cont}} \boldsymbol{\beta}} c_{\boldsymbol{\delta}}\ [T, C]_{\text{per}}(\partial_X, \partial_Y)\ \Delta_{L[\boldsymbol{\delta}]}, \end{aligned} \tag{91}$$

where $\boldsymbol{\beta} = (\overline{3}, \overline{1}, \underline{0}, \underline{2}, \underline{3}, \underline{5})$. For γ associated to the $\boldsymbol{\alpha}$ above, $m_2 + p_2 + 1 = 3$. In addition

$$[T, C]_{\text{per}}(X_6, Y_6) = \sum_{\sigma \in S_{\{1,2\}} \times S_{\{3,5\}} \times S_{\{4,6\}}} z_{\sigma(1)}^{\overline{1}} z_{\sigma(2)}^{\underline{0}} z_{\sigma(3)}^{\underline{0}} z_{\sigma(4)}^{\underline{2}} z_{\sigma(5)}^{\underline{1}} z_{\sigma(6)}^{\underline{2}}. \tag{92}$$

It follows then that

$$[T, C]_{\text{per}}(\partial_{X_6}, \partial_{Y_6}) = (\partial_{y_1} + \partial_{y_2})(\partial_{x_3} + \partial_{x_5})(2 \cdot \partial_{x_4}^2 \partial_{x_6}^2). \tag{93}$$

So, by Lemma 27, $[T, C]_{\text{per}}(\partial_{X_6}, \partial_{Y_6})\Delta_{L[\boldsymbol{\beta}]}$ can be expanded as

$$\mathrm{sgn}(\iota_{T,U}) \sum_{\phi \in S_6} \mathrm{sgn}(\phi)\, d_\phi [T^t, (E_\phi^{\boldsymbol{\beta}})^t]_{\det}. \tag{94}$$

It is easily checked for this particular example that the entries of $E_\phi^{\boldsymbol{\beta}}$ will all be in $\mathcal{A}$ only if $\phi \in S_{1,2,3} \times S_{4,5,6}$ with $\phi(1) \neq 3$. There are 24 such permutations. Table 1 gives these ϕ along with the coefficients d_ϕ and diagrams $E_\phi^{\boldsymbol{\beta}}$. We leave the expansion of the sum in (91) to the reader. This completes our example.

	ϕ	d_ϕ	$(E_\phi^{\boldsymbol{\beta}})^t$	ϕ	d_ϕ	$(E_\phi^{\boldsymbol{\beta}})^t$
Cases	ε	720	$\begin{array}{\|c\|c\|c\|}\hline \underline{0}&\underline{1}&\underline{3}\\\hline \bar{2}&\bar{1}&\underline{1}\\\hline\end{array}$	$(5,6)$	720	$\begin{array}{\|c\|c\|c\|}\hline \underline{0}&\underline{1}&\underline{1}\\\hline \bar{2}&\bar{1}&\underline{3}\\\hline\end{array}$
1 & 2	$(2,3)$	720	$\begin{array}{\|c\|c\|c\|}\hline \bar{1}&\underline{1}&\underline{3}\\\hline \bar{2}&\underline{0}&\underline{1}\\\hline\end{array}$	$(2,3)(5,6)$	720	$\begin{array}{\|c\|c\|c\|}\hline \bar{1}&\underline{1}&\underline{1}\\\hline \bar{2}&\underline{0}&\underline{3}\\\hline\end{array}$
Case 3	$(4,5)$	360	$\begin{array}{\|c\|c\|c\|}\hline \underline{0}&\underline{2}&\underline{3}\\\hline \bar{2}&\bar{1}&\underline{0}\\\hline\end{array}$	$(4,6)$	120	$\begin{array}{\|c\|c\|c\|}\hline \underline{0}&\underline{4}&\underline{0}\\\hline \bar{2}&\bar{1}&\underline{1}\\\hline\end{array}$
	$(2,3)(4,5)$	360	$\begin{array}{\|c\|c\|c\|}\hline \bar{1}&\underline{2}&\underline{3}\\\hline \bar{2}&\underline{0}&\underline{0}\\\hline\end{array}$	$(2,3)(4,6)$	120	$\begin{array}{\|c\|c\|c\|}\hline \bar{1}&\underline{4}&\underline{0}\\\hline \bar{2}&\underline{0}&\underline{1}\\\hline\end{array}$
	$(4,6,5)$	180	$\begin{array}{\|c\|c\|c\|}\hline \underline{0}&\underline{4}&\underline{1}\\\hline \bar{2}&\bar{1}&\underline{0}\\\hline\end{array}$	$(4,5,6)$	360	$\begin{array}{\|c\|c\|c\|}\hline \underline{0}&\underline{2}&\underline{0}\\\hline \bar{2}&\bar{1}&\underline{3}\\\hline\end{array}$
	$(2,3)(4,6,5)$	180	$\begin{array}{\|c\|c\|c\|}\hline \bar{1}&\underline{4}&\underline{1}\\\hline \bar{2}&\underline{0}&\underline{0}\\\hline\end{array}$	$(2,3)(4,5,6)$	360	$\begin{array}{\|c\|c\|c\|}\hline \bar{1}&\underline{2}&\underline{0}\\\hline \bar{2}&\underline{0}&\underline{3}\\\hline\end{array}$
	$(1,2,3)$	240	$\begin{array}{\|c\|c\|c\|}\hline \bar{3}&\underline{1}&\underline{3}\\\hline \underline{0}&\underline{0}&\underline{1}\\\hline\end{array}$	$(1,2,3)(4,5)$	120	$\begin{array}{\|c\|c\|c\|}\hline \bar{3}&\underline{2}&\underline{3}\\\hline \underline{0}&\underline{0}&\underline{0}\\\hline\end{array}$
	$(1,2,3)(4,6)$	60	$\begin{array}{\|c\|c\|c\|}\hline \bar{3}&\underline{4}&\underline{0}\\\hline \underline{0}&\underline{0}&\underline{1}\\\hline\end{array}$	$(1,2,3)(5,6)$	240	$\begin{array}{\|c\|c\|c\|}\hline \bar{3}&\underline{1}&\underline{1}\\\hline \underline{0}&\underline{0}&\underline{3}\\\hline\end{array}$
	$(1,2,3)(4,6,5)$	60	$\begin{array}{\|c\|c\|c\|}\hline \bar{3}&\underline{4}&\underline{1}\\\hline \underline{0}&\underline{0}&\underline{0}\\\hline\end{array}$	$(1,2,3)(4,5,6)$	120	$\begin{array}{\|c\|c\|c\|}\hline \bar{3}&\underline{2}&\underline{0}\\\hline \underline{0}&\underline{0}&\underline{3}\\\hline\end{array}$
	$(1,2)$	0	$\begin{array}{\|c\|c\|c\|}\hline \underline{0}&\underline{1}&\underline{3}\\\hline \underline{0}&\bar{3}&\underline{1}\\\hline\end{array}$	$(1,2)(4,5)$	0	$\begin{array}{\|c\|c\|c\|}\hline \underline{0}&\underline{2}&\underline{3}\\\hline \underline{0}&\bar{3}&\underline{0}\\\hline\end{array}$
	$(1,2)(4,6)$	0	$\begin{array}{\|c\|c\|c\|}\hline \underline{0}&\underline{4}&\underline{0}\\\hline \underline{0}&\bar{3}&\underline{1}\\\hline\end{array}$	$(1,2)(5,6)$	0	$\begin{array}{\|c\|c\|c\|}\hline \underline{0}&\underline{1}&\underline{1}\\\hline \underline{0}&\bar{3}&\underline{3}\\\hline\end{array}$
	$(1,2)(4,6,5)$	0	$\begin{array}{\|c\|c\|c\|}\hline \underline{0}&\underline{4}&\underline{1}\\\hline \underline{0}&\bar{3}&\underline{0}\\\hline\end{array}$	$(1,2)(4,5,6)$	0	$\begin{array}{\|c\|c\|c\|}\hline \underline{0}&\underline{2}&\underline{0}\\\hline \underline{0}&\bar{3}&\underline{3}\\\hline\end{array}$

TABLE 1. Expansion of Example 30

Proof. Let $T, U \in SYT_n$. Using Theorem 25, we have that

$$\begin{aligned}
&h_q(\partial_X)h_{q'}(\partial_Y)\ [T,C]_{\mathrm{per}}(\partial_X,\partial_Y)\ \Delta_\gamma(X,Y)\\
&\qquad = [T,C]_{\mathrm{per}}(\partial_X,\partial_Y) h_q(\partial_X)h_{q'}(\partial_Y)\ \Delta_\gamma(X,Y)\\
&\qquad = [T,C]_{\mathrm{per}}(\partial_X,\partial_Y)\left(c_{\boldsymbol{\beta}}\ \Delta_{L[\boldsymbol{\beta}]} + \sum_{\boldsymbol{\delta} >_{\mathrm{cont}} \boldsymbol{\beta}} c_{\boldsymbol{\delta}}\ \Delta_{L[\boldsymbol{\delta}]}\right) \qquad (95)\\
&\qquad = c_{\boldsymbol{\beta}}\ [T,C]_{\mathrm{per}}(\partial_X,\partial_Y)\ \Delta_{L[\boldsymbol{\beta}]} + \sum_{\boldsymbol{\delta} >_{\mathrm{cont}} \boldsymbol{\beta}} c_{\boldsymbol{\delta}}\ [T,C]_{\mathrm{per}}(\partial_X,\partial_Y)\ \Delta_{L[\boldsymbol{\delta}]},
\end{aligned}$$

with $c_{\boldsymbol{\beta}} > 0$. Now (abbreviating $\iota_{T,U}$ by ι), Lemma 27 implies that

$$[T, C]_{\mathrm{per}}(\partial_X, \partial_Y)\ \Delta_{L[\boldsymbol{\beta}]} = \mathrm{sgn}(\iota) \sum_{\phi \in S_n} \mathrm{sgn}(\phi)\ d_\phi\ [T^t, (E_\phi^{\boldsymbol{\beta}})^t]_{\mathrm{det}}. \tag{96}$$

Therefore,

$$\begin{aligned} &h_q(\partial_X)\, h_{q'}(\partial_Y)[T, C]_{\mathrm{per}}(\partial_X, \partial_Y)\ \Delta_\gamma(X, Y) \\ &\quad = \mathrm{sgn}(\iota) \sum_{\phi \in S_n} \mathrm{sgn}(\phi)\ \left(c_{\boldsymbol{\beta}}\ d_\phi\ [T^t, (E_\phi^{\boldsymbol{\beta}})^t]_{\mathrm{det}} + \sum_{\delta >_{\mathrm{cont}} \beta} c_{\boldsymbol{\delta}}\ d_\phi\ [T^t, (E_\phi^{\boldsymbol{\delta}})^t]_{\mathrm{det}} \right). \end{aligned} \tag{97}$$

Our goal is to show that there is a minimum bitableau (with respect to $<_{\mathrm{bitab}}$) occurring on the right-hand side of (97). First note that many of the bitableaux $(T^t, (E_\phi^{\boldsymbol{\beta}})^t)$ and $(T^t, (E_\phi^{\boldsymbol{\delta}})^t)$ are not standard. However, Theorem 7 tells us how the shapes and column sequences are affected by straightening.

We split into three cases dependent on the indexing permutations ϕ. (A proof of an argument with similar statements and complete details can be in found in [3, Theorem 6.2].)

Case 1: $\phi = \varepsilon$.
We get the first term on the right-hand side of (90) with $d = c_{\boldsymbol{\beta}}\, d_\varepsilon > 0$.

Case 2: $\phi \neq \varepsilon$, but $\phi(z_1^{c_1^T} \cdots z_n^{c_n^T}) = z_1^{c_1^T} \cdots z_n^{c_n^T}$.
For such ϕ, $\kappa(E_\phi^{\boldsymbol{\beta}}) = \kappa(P)$. However, it follows from the construction of $C = C_\gamma(U)$ from U that $\mathrm{rs}((E_\phi^{\boldsymbol{\beta}})^t) = \mathrm{cs}(E_\phi^{\boldsymbol{\beta}}) >_{\mathrm{lex}(\mathcal{A})} \mathrm{cs}(P) = \mathrm{rs}(P^t)$. Note that the $E_\phi^{\boldsymbol{\beta}}$ arising in this case are standard and do not need to be straightened.

Case 3: $\phi(z_1^{c_1^T} \cdots z_n^{c_n^T}) \neq z_1^{c_1^T} \cdots z_n^{c_n^T}$.
We must have $\mathrm{con}_{\mathrm{X}}(E_\phi^{\boldsymbol{\beta}}) <_{\mathrm{lex}} \mathrm{con}_{\mathrm{X}}(P)$ or $\mathrm{con}_{\mathrm{Y}}(E_\phi^{\boldsymbol{\beta}}) >_{\mathrm{lex}} \mathrm{con}_{\mathrm{Y}}(P)$.

It is not difficult to see that the $E_\phi^{\boldsymbol{\delta}}$ with $\boldsymbol{\delta} >_{\mathrm{cont}} \boldsymbol{\beta}$ that

$$(T, E_\phi^{\boldsymbol{\delta}}) >_{\mathrm{bitab}} (T, P). \tag{98}$$

Thus, combinations of ϕ and $\boldsymbol{\delta} \neq \boldsymbol{\beta}$ substituted into equation (97) (and using Theorem 7) yield bideterminants $[S, W]_{\mathrm{det}}$ with $(S, W) >_{\mathrm{bitab}} (T, P)$. Thus, we have

$$\begin{aligned} &h_q(\partial_X) h_{q'}(\partial_Y)\ [T, C]_{\mathrm{per}}(\partial_X, \partial_Y)\ \Delta_\gamma(X, Y) = \\ &\qquad\qquad d\ [T^t, P^t]_{\mathrm{det}} + \sum_{(S,W) >_{\mathrm{bitab}} (T,P)} d_{S,W} [S^t, W^t]_{\mathrm{det}}, \end{aligned} \tag{99}$$

with $d \neq 0$. This proves the theorem. $\square$

Since we have that $\mathcal{J}_\gamma(X_n, Y_n) \subset \mathcal{I}_\gamma(X_n, Y_n)$, the fact that the Hilbert series in equation (72) equals the summation in equation (76) implies that we must have $\mathcal{J}_\gamma(X_n, Y_n) = \mathcal{I}_\gamma(X_n, Y_n)$. Thus, equation (72) must give the Hilbert series for $\mathbb{C}[X_n, Y_n]_{\mathcal{I}_\gamma}$. Since the collection $\mathcal{BB}_\gamma$ is linearly independent in $\mathbb{C}[X_n, Y_n]_{\mathcal{I}_\gamma}$ and it gives the correct Hilbert series, we must have that $\mathcal{BB}_\gamma$ is in fact a basis for $\mathbb{C}[X_n, Y_n]_{\mathcal{I}_\gamma}$. The proof of Theorem 26 is now complete.

9. Some notes, applications and conjectures

Remark 31. Suppose $\mathcal{D}$ is any basis for $\mathbb{C}[X_n, Y_n]_{\mathcal{I}_\gamma}$ where $\gamma = ((m_1, n - m_1 + 1), (0,0), (0,0))$ for some $m_1 > 0$. The basis $\mathcal{D}$ can be substituted in the place of $\mathcal{B}_\gamma$ in the definition of $\mathcal{BB}_\gamma$ of equation (74) such that $\mathcal{BB}_\gamma$ still yields a basis for $\mathbb{C}[X_n, Y_n]_{\mathcal{I}_\gamma}$. Examples of such bases $\mathcal{D}$ include descent monomials (see [3], [8] or [16]), Artin monomials (see [5]), Schubert monomials (see [13]) and Higher Specht Polynomials (see [15]).

Remark 32. The ideas of the previous sections are easily extendable to the complex reflection groups $G(r,p,n)$. In this case, the lattice diagrams utilized are those of the form $L[\boldsymbol{\beta}]$ where $L[\boldsymbol{\alpha}]$ is a hollow lattice diagram and $\boldsymbol{\beta} = \{(r\alpha_{i,1}, r\alpha_{i,2}) : \alpha \in \boldsymbol{\alpha}\}$. The resulting bases give representations of the complex reflection groups by ways of m-tableaux, standard m-tableaux and cocharge m-tableaux. See [3] to see how this translation is accomplished.

Remark 33. Some of the results of this paper can be extended to diagonally symmetric and anti-symmetric rings in the four sets of variables X_n, Y_n, Z_n and W_n. The diagonal action of S_n on $\mathbb{C}[X_n, Y_n, Z_n, W_n]$ and the rings of symmetric polynomials and anti-symmetric polynomials $\mathbb{C}^+[X_n, Y_n, Z_n, W_n]$ and $\mathbb{C}^-[X_n, Y_n, Z_n, W_n]$ are defined in the natural manner. For $\sigma \in S_n$, set $\varepsilon^+(\sigma) = 1$, $\varepsilon^-(\sigma) = \mathrm{sgn}(\sigma)$. For each of $+$ and $-$, define

$$\begin{aligned}(100)\quad R^{\pm}_{\gamma_1,\gamma_2} = \mathbb{C}^{\pm}[X_n, Y_n, Z_n, W_n]/\big\{&P \in \mathbb{C}^{\pm}[X_n, Y_n, Z_n, W_n] : \\ &P(\partial_X, \partial_Y, \partial_Z, \partial_W)\ \Delta_{\gamma_1}(X_n, Y_n)\ \Delta_{\gamma_2}(Z_n, W_n) = 0\big\}.\end{aligned}$$

Furthermore, for an arbitrary standard tableau Q, set

$$(101)\quad [T_1, T_2]^+_{\mathrm{per}} = \sum_{\sigma \in S_n} \sigma\left([Q, T_1]_{\mathrm{per}}(X_n, Y_n)\ [Q, T_2]_{\mathrm{per}}(Z_n, W_n)\right)$$

$$(102)\quad [T_1, T_2]^-_{\mathrm{per}} = \sum_{\sigma \in S_n} \sigma\left([Q, T_1]_{\mathrm{per}}(X_n, Y_n)\ [Q, T_2]_{\mathrm{per}}(\partial_Z, \partial_W)\Delta_{\gamma_2}(Z_n, W_n)\right).$$

Using the techniques found in [4], it is not difficult to show the following theorem:

Theorem 34. *For each of $+$ and $-$,*

$$\begin{aligned}(103)\quad \mathcal{BB}^{\pm}_{\gamma_1,\gamma_2|} = \Big\{&h_{q_1,q_2,\ldots,q_{p_1+1}}(X,Y)\ h_{q'_1,q'_2,\ldots,q'_{p_1+1}}(Z,W)\ [T_1,T_2]^{\pm}_{per} : \\ &\mathrm{sh}(T_1) = \mathrm{sh}(T_2), T_1 \in \mathcal{CO}_{n,\gamma_1}, T_2 \in \mathcal{CO}_{n,\gamma_2}\big\},\end{aligned}$$

where $(q_1, q_2, \ldots, q_{p_1+1}) \in Q_{k_1-1,p_1+1}$ and $(q'_1, q'_2, \ldots, q'_{p_2+1}) \in Q_{k_2-1,p_1+1}$, is a basis for $R^{\pm}_{\gamma_1,\gamma_2}$.

These are generalizations of rings that have been studied in [2, 16], for example. Furthermore, it should be noted that this theorem is not a generalization of the theorems found in [4]. Rather, we are just noting that the techniques found there easily generalize to the situation in this paper.

References

[1] Allen, E. E. A conjecture of Procesi and a new basis for the decomposition of the graded left regular representation of S_n. *Adv. Math. 100*, 2 (1993), 262–292.

[2] Allen, E. E. The descent monomials and a basis for the diagonally symmetric polynomials. *J. Algebraic Combin. 3*, 1 (1994), 5–16.

[3] Allen, E. E. Some graded representations of the complex reflection groups. *J. Combin. Theory Ser. A 87*, 2 (1999), 287–332.

[4] Allen, E. E. Bitableaux bases for some Garsia-Haiman modules and other related modules. *Electron. J. Combin. 9*, 1 (2002), Research Paper 36, 59 pp. (electronic).

[5] Artin, E. *Galois Theory*, second ed. Notre Dame Mathematical Lectures, no. 2. University of Notre Dame, Notre Dame, Ind., 1944.

[6] Bergeron, F., Garsia, A., and Tesler, G. Multiple left regular representations generated by alternants. *J. Combin. Theory Ser. A 91*, 1-2 (2000), 49–83. In memory of Gian-Carlo Rota.

[7] Désarménien, J., Kung, J. P. S., and Rota, G.-C. Invariant theory, Young bitableaux, and combinatorics. *Advances in Math. 27*, 1 (1978), 63–92.

[8] Garsia, A. M. Combinatorial methods in the theory of Cohen-Macaulay rings. *Adv. in Math. 38*, 3 (1980), 229–266.

[9] Garsia, A. M., and Haiman, M. A graded representation model for Macdonald's polynomials. *Proc. Nat. Acad. Sci. U.S.A. 90*, 8 (1993), 3607–3610.

[10] Grosshans, F. D., Rota, G.-C., and Stein, J. A. *Invariant theory and superalgebras*, vol. 69 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1987.

[11] Haglund, J., Haiman, M., Loehr, N., Remmel, J. B., and Ulyanov, A. A combinatorial formula for the character of the diagonal coinvariants. *Duke Math. J. 126*, 2 (2005), 195–232.

[12] Haiman, M. Combinatorics, symmetric functions, and Hilbert schemes. In *Current developments in mathematics, 2002*. Int. Press, Somerville, MA, 2003, pp. 39–111.

[13] Lascoux, A., and Schützenberger, M.-P. Polynômes de Schubert. *C. R. Acad. Sci. Paris Sér. I Math. 294*, 13 (1982), 447–450.

[14] Macdonald, I. G. *Symmetric functions and Hall polynomials*, second ed. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.

[15] Morita, H., and Yamada, H.-F. Higher Specht polynomials for the complex reflection group $G(r, p, n)$. *Hokkaido Math. J. 27*, 3 (1998), 505–515.

[16] Reiner, V. Quotients of Coxeter complexes and P-partitions. *Mem. Amer. Math. Soc. 95*, 460 (1992), vi+134.

[17] Stanley, R. P. Invariants of finite groups and their applications to combinatorics. *Bull. Amer. Math. Soc. (N.S.) 1*, 3 (1979), 475–511.

[18] Stembridge, J. R. On the eigenvalues of representations of reflection groups and wreath products. *Pacific J. Math. 140*, 2 (1989), 353–396.

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